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Semiregular Modules with Respect to a Fully Invariant Submodule#

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ABSTRACT

Let M be a left R-module and F a submodule of M for any ring R. We call M *F-semiregular* if for every $x \in M$, there exists a decomposition $M = A \oplus B$ such that A is projective, $A \leq Rx$ and $Rx \cap B \leq F$. This definition extends several notions in the literature. We investigate some equivalent conditions to F-semiregular modules and consider some certain fully invariant submodules such as $Z(M), Soc(M), \delta(M)$. We prove, among others, that if M is a finitely generated projective module, then M is quasi-injective if and only if M is $Z(M)$-semiregular and $M \oplus M$ is CS. If M is projective $Soc(M)$-semiregular module, then M is semiregular. We also characterize QF-rings R with $J(R)^2 = 0$.

Key Words: Semiregular modules; CS modules; Quasi-injective modules; ACS rings; QF rings.

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1. INTRODUCTION

Perfect, semiperfect and semiregular (or f-semiperfect) rings constiture the class of rings that posses beautiful homological and non homological properties. The concept of semiperfect rings has been generalized to semiperfect modules by Mares (1963). Mares calls a module M *a semiperfect module* if every quotient of M has a projective cover. Nicholson (1976) proves that a projective module M is semiperfect if and only if it is semiregular, $Rad(M) \ll M$ and $M/Rad(M)$ is semisimple. Semiregular modules are known as a unified generalization of semiperfect modules and regular modules of Zelmanowitz. There has been a great deal of work on semiregular modules by several authors (e.g., Azumaya, 1991; Nicholson, 1976; Wisbauer, 1991; Xue, 1995).

Zhou (2000) defines δ-semiregular and δ-semiperfect rings as a generalization of semiregular and semiperfect rings. On the other hand, Nicholson and Yousif (2001) consider I-semiregular rings for an ideal I of a ring R and study $Z({}_RR)$-semiregular rings. Now in this paper, we define F-semiregular modules M for a submodule F of a module M and consider some certain fully invariant submodules such as $Z(M)$, $Soc(M), \delta(M)$ (is defined in Zhou, 2000).

If M is semiregular, then for every $x \in M$ there exists a decomposition $M = A \oplus B$ such that $A \leq Rx$ is projective and $B \cap Rx \ll M$ or equivalently $B \cap Rx \leq Rad(M)$. Therefore, here we may consider any (fully invariant) submodule F or M instead of $Rad(M)$, and we denote such modules as F-semiregular modules. In Sec. 2, we investigate the equivalent conditions to F-semiregular modules inspired by Nicholson and Yousif's results. Some of their results are directly generalized but some are not, and we define (S_1) and (S_2) properties for them.

In Sec. 3, we consider $Z(\cdot)$-semiregular modules. We prove that for a finitely generated projective module M, M is quasi-injective if and only if M is $Z(M)$-semiregular and $M \oplus M$ is CS.

In the last section, we consider $Soc(\cdot)$-semiregular and $\delta(\cdot)$-semiregular modules and investigate the relationship between them. We prove that if M is a countably generated $\delta(M)$-semiregular module with $\delta(M) \ll_\delta M$ then M is isomorphic to a direct sum of projective cyclic submodules of M. Any projective $Soc(M)$-semiregular module M is semiregular. And we characterize left Artinian rings R with $J(R)^2 = 0$ and quasi-Frobenius (QF) rings R with $J(R)^2 = 0$. At the end of the paper, we give some counter examples.

Throughout this paper, R denotes an associative ring with identity and modules M are unitary left R-modules. For a module $M, Rad(M)$ and $Z(M)$ are the Jacobson radical and the singular submodule of M. We write $J(R)$ for the Jacobson radical of R. The dual of M is denoted by $M^* = Hom_R(M, R)$. A submodule N of M is called *small* in M, denoted by $N \ll M$, whenever for any submodule L of $M, N + L = M$ implies $L = M$. Dually we use $N \leq_e M$ to signify that N is an essential submodule of M. For a direct summand K of M we write $K \leq^\oplus M$.

A submodule N of a module M is said to *lie over a summand* of M if there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $B \cap N$ is small in M. An element x in M is called *regular* if $(x\alpha)\ x = x$ for some $\alpha \in M^*$. Zelmanowitz (1973) calls a module *regular* if each of its elements is regular, equivalently if every finitely generated submodule is a projective summand. Nicholson (1976) calls an element

x and M *semiregular* if Rx lies over a projective summand of M. A module called *semiregular* if each of its elements is semiregular.

2. F-SEMIREGULAR MODULES

In this chapter, we investigate some equivalent conditions to F-semiregular modules.

Definition 2.1. Let F be a submodule of an R-module M. An element x in M is said to be *F-semiregular* in M if there exists a decomposition $M = A \oplus B$ such that A is projective, $A \leq Rx$ and $Rx \cap B \leq F$. A module M is called *an F-semiregular module* if every elements x in M is F-semiregular.

Clearly the class of F-semiregular modules contains all regular modules. Also M is semiregular if and only if M is $Rad(M)$-semiregular. If M is semiregular and F is a submodule of M such that $Rad(M) \leq F$ then M is F-semiregular. For $M = R$ and an ideal $F = I$, I-semiregularity of rings is defined by Nicholson and Yousif (2001). Now we consider the module theoretic version of some results of Nicholson and Yousif.

Proposition 2.2. *Let F be a submodule of a module M. Then the following conditions are equivalent for $x \in M$.*

(1) *x is F-semiregular.*
(2) *There exists $\alpha \in M^*$ such that $(x\alpha)^2 = x\alpha$ and $x - (x\alpha)x \in F$.*
(3) *There exists a homomorphism γ from M to Rx such that $\gamma^2 = \gamma, M\gamma$ is projective and $x - x\gamma \in F$.*

When these conditions hold we have

(4) *There exists a regular element $y \in Rx$ such that $x - y \in F$ and $Rx = Ry(x - y)$. If F is fully invariant then* (1)–(3) *are equivalent to* (4).

Proof. (1) $\Rightarrow$ (2). Suppose for x in M there exists a decomposition $M = A \oplus B$ such that A is projective, $A \leq Rx$ and $Rx \cap B \leq F$. Then there exist $x_i \in A$ and $\alpha_i \in A^* = Hom_R(A, R)$ $(i = 1, \ldots, n)$ such that $y = \sum_{i=1}^{n}(y\alpha_i)x_i$ for any $y \in A$. Hence α_i extends to M by $(a + b)\beta_i = a\alpha_i$. Write $x_i = r_i x$ with $r_i \in R$ and let $\alpha = \sum \beta_i r_i$. Then $\alpha \in M^*$. Write $x = a + b$ with $a \in A, b \in B$. We get $(x\alpha)x = \sum(x\beta_i)r_i x = \sum(a\alpha_i)x_i = a$. Therefore, $x - a = x - (x\alpha)x = b \in Rx \cap B \leq F$.

(2) $\Rightarrow$ (3). Let x and α be as in (2) and let $y = (x\alpha)x$. Then $y = (y\alpha)y$. By Nicholson (1976, Lemma 1.1), Ry is a projective submodule of Rx and $M = Ry \oplus W$ where $W = \{w \in M : (w\alpha)y = 0\}$. Let $\gamma : M \to Ry$ be the projection map. Hence it is sufficient to show that $x - x\gamma \in F$. Write $x = ry + w \in M$ where $r \in R$ and $w \in W$. Then $0 = (x - ry)\alpha y = (x\alpha)y - r(y\alpha)y = (x\alpha)y - ry$, so $x\gamma = ry = (x\alpha)y = y$. Therefore, $x - x\gamma = x - y \in F$.

(3) $\Rightarrow$ (1). Suppose (3) holds. Then $M = M\gamma \oplus M(1 - \gamma)$ and $Rx \cap M(1 - \gamma) = Rx(1 - \gamma) \leq F$.

$(2) \Rightarrow (4)$. Let x, α, y and W be as in $(2) \Rightarrow (3)$. Then $W \cap Rx = R(x-y)$. Therefore, $Rx = Ry \oplus R(x-y)$.

$(4) \Rightarrow (1)$. Assume F is fully invariant. Let x and y be as in (4) and let $\alpha \in M^*$ be such that $(y\alpha)y = y$. Then $M = Ry \oplus W$ where $W = \{w \in M : (w\alpha)y = 0\}$. Hence, $Rx = Ry \oplus (Rx \cap W)$. Let $\pi : M \to W$ be the projection map. Then $Rx \cap W = (Rx \cap W)\pi = (Rx)\pi = (R(x-y))\pi \leq (F)\pi \leq F$. This completes the proof. □

Taking $M = R$ and $F = I$ an ideal of R yields (Nicholson and Yousif, 2001, Lemma 1.1). Our next results gives the characterization of F-semiregular modules.

Theorem 2.3. *Let F be a fully invariant submodule of a module M. Then the following conditions are equivalent.*

(1) *M is F-semiregular.*
(2) *For any finitely generated submodule N of M, there exists a homomorphism γ from M to N such that $\gamma^2 = \gamma$, $M\gamma$ is projective and $N(1-\gamma) \leq F$.*
(3) *For any finitely generated submodule N of M, there exists a decomposition $M = A \oplus B$ such that A is a projective submodule of N and $N \cap B \leq F$.*
(4) *For any finitely generated submodule N of M, N can be written as $N = A \oplus S$ where A is a projective summand of M and $S \leq F$.*

When these conditions hold we have

(5) *For all $x \in M$, there exists a regular element $y \in M$ such that $x - y \in F$.*
(6) *Every submodule of M that is not contained in F contains a regular element not in F.*
(7) *$Rad(M) \leq F$ and $Z(M) \leq F$.*

Proof. $(1) \Rightarrow (2)$. Let N be a finitely generated submodule with generators $x_0, \ldots, x_n$. We use the induction on the generating set. By assumption choose $\beta : M \to Rx_n$ such that $\beta^2 = \beta$, $M\beta$ is projective and $(x_n)(1-\beta) \in F$. Set $K = Rx_0(1-\beta) + \cdots + Rx_{n-1}(1-\beta)$ and by induction choose $\alpha : M \to K$ such that $\alpha^2 = \alpha$, $M\alpha$ is projective and $K(1-\alpha) \leq F$. Define $\gamma = \beta + \alpha - \beta\alpha$. Then $\gamma = \gamma^2$ and $M\gamma = M\beta \oplus M\alpha$ since $\alpha\beta = 0$. Hence $M\gamma$ is projective. It is enough to show that $N(1-\gamma) \leq F$. Since $N = K + Rx_n$ it follows that $M\gamma = M\beta + M\alpha \leq Rx_n + K = N$. Take $n = a + rx_n \in N$ as $a \in K$ and $rx_n \in K$ and $rx_n \in Rx_n$. $(a + rx_n)(1-\gamma) = (a + rx_n)(1-\beta)(1-\alpha) = (a(1-\beta) + rx_n(1-\beta))(1-\alpha) = a(1-\alpha) + (rx_n(1-\beta))(1-\alpha) \in F$. Therefore $N(1-\gamma) \leq F$.

$(2) \Rightarrow (3)$. Let N and γ be as in (2). Then $N \cap (M)(1-\gamma) = N(1-\gamma)$. Hence, $M = M\gamma \oplus M(1-\gamma)$, $M\gamma$ is projective and $N \cap (M)(1-\gamma) = N(1-\gamma) \leq F$.

$(3) \Rightarrow (2)$. Let N be a finitely generated submodule of M. By (3), $M = A \oplus B$ where A is a projective submodule of N and $N \cap B \leq F$. Then $N = A \oplus (B \cap N)$. Now consider the projection map $\pi : M \to A$. Let $\gamma = \pi i$ where i is the inclusion map from A to N. Then $\gamma^2 = \gamma$, $M\gamma = A$ is projective and $N(1-\gamma) \leq F$.

$(3) \Rightarrow (4)$. It is clear.

(4) $\Rightarrow$ (1). Let N be a cyclic submodule of M. Then $N = A \oplus S$ with A a projective summand of M and $S \leq F$. Then $M = A \oplus B$ for some B. Let $\pi : M \to B$ be the projection map. Then $N = A \oplus (N \cap B)$ and $N \cap B = (N)\pi = (S)\pi \leq (F)\pi \leq F$.

(1) $\Rightarrow$ (5) and (1) $\Rightarrow$ (6) are by Proposition 2.2(4).

(1) $\Rightarrow$ (7). Note that every cyclic submodule of $Rad\, M$ is small in M and every projective singular module is a zero module, so (7) follows from (6) and (Nicholson, 1976, Lemma 1.1). □

Observe that (2) $\Leftrightarrow$ (3) $\Rightarrow$ (1) holds for any submodule F of a module M.

Note that if I is an ideal of a ring R then IM is a fully invariant submodule of M.

Theorem 1.2 in Nicholson and Yousif (2001) follows from Theorem 2.3 by taking $M = R$ and $F = IM$.

Nicholson and Yousif (2001) give a counter example showing that condition (5) in Theorem 2.3 does not imply I-semiregularity by taking $M = R = \mathbb{Z}$ and $I = 2\mathbb{Z}$. In Theorem 2.6, we give the equivalence under some conditions. First we give some definitions.

Zhou (2000) defines that a submodule N of a module M is called δ-*small* in M if $N + K \neq M$ for any proper submodule K of M/K singular, denoted by $N \ll_\delta M$.

Lemma 2.4 (Zhou, 2000, Lemma 1.2). *Let N be a submodule of a module M. Then $N \ll_\delta M$ if and only if $M = X \oplus Y$ for a projective semisimple submodule Y with $Y \leq N$ whenever $X + N = M$.*

Also Zhou introduces the following fully invariant submodule of a module M.

$$\delta(M) = \bigcap\{N \leq M : M/N \text{ is singular simple}\}.$$

Then $\delta(M)$ is the sum of all δ-small submodules of M by Zhou (2000, Lemma 1.5), and hence $Rad(M) \leq \delta(M)$. If every proper submodule of M is contained in a maximal submodule of M, then $\delta(M) \ll_\delta M$.

Let F be a submodule of a module M. Then F is said to satisfy

- (R_1) If for every summand A of M, $A \cap F$ lies over a summand of M.
- (R_2) If for every regular element y in M, $Ry \cap F$ lies over a summand of M.
- (S_1) If for every summand N of M, there exists a decomposition $M = A \oplus B$ such that $A \leq N \cap F$ and $B \cap N \cap F \ll_\delta M$.
- (S_2) If for every regular element y in M, there exists a decomposition $M = A \oplus B$ such that $A \leq Ry \cap F$ and $B \cap Ry \cap F \ll_\delta M$.

Clearly $(R_1) \Rightarrow (R_2)$ and $(S_1) \Rightarrow (S_2)$. For $M = R, (R_1) \Leftrightarrow (R_2)$ and $(S_1) \Leftrightarrow (S_2)$. If $F \leq \delta(M)$ then $Ry \cap F \leq Ry \cap \delta(M) = \delta(Ry) \ll_\delta M$ for any regular element $y \in M$. Hence F satisfies (S_2). If $F \ll_\delta M$, then F satisfies (S_1). We also have the following diagram.

$$\begin{array}{ccc} (R_1) & \Longrightarrow & (R_2) \\ \Downarrow & & \Downarrow \\ (S_1) & \Longrightarrow & (S_2) \end{array}$$

In general (S_1) does not imply (R_1) and (S_2) does not imply (R_2).

Example 2.5. Let T be the infinite product of F_i, where each $F_i = \mathbb{Z}_2$ and let R be the subring of T generated by $\bigoplus_{i\geq 1} F_i$ and the identity of T. Then $\delta({}_RR) = Soc({}_RR)$ satisfies (S_1) but not (R_2).

Theorem 2.6. *Let F be a fully invariant submodule of a module M and satisfy (S_2). Let $x \in M$. If there exists a regular element $y \in M$ such that $x - y \in F$, then x is F-semiregular.*

Proof. Let $x \in M$. By assumption there exists a regular element $y \in M$ such that $x - y \in F$ and there is a decomposition $M = K \oplus L$ such that $K \leq F \cap Ry$ and $F \cap Ry \cap L \ll_\delta M$. Since y is regular we have $M = Ry \oplus W$ for a submodule W of M and Ry is projective. It follows that $M = (Ry \cap L) \oplus K \oplus W$ and $F = (Ry \cap L \cap F) \oplus K \oplus (W \cap F)$ as F is fully invariant. On the other hand, $F \cap Ry \cap L \ll_\delta Ry + F = Rx + F$ as $x - y \in F$ and $Ry \leq^\oplus M$. Then, by Lemma 2.4, $Rx + F = (Rx + K + (W \cap F)) \oplus D$ for a projective semisimple submodule D of $F \cap Ry \cap L$. Then $Ry \cap L = E \oplus D$ where $E = (Ry \cap L) \cap (Rx + K + (W \cap F))$.

Let π be the projection map from M to E. Then $E = (Ry + F)\pi = (Rx + F)\pi = (Rx)\pi$. Since $\alpha := \pi|_{Rx}$ is an epimorphism and E is projective, α splits. Then there exists $\pi' : E \to Rx$ such that $\pi'\alpha = 1$ and $Rx = Im\pi' \oplus Ker(\alpha)$. Let $A := Im\pi'$. Since $Ker(\alpha) \cap A = 0$ and $A \leq Rx$, $Ker(\pi) \cap A = 0$. Also $(A)\pi = E$. Hence $\pi|_A$ is an isomorphism. By Proposition 5.5 in Anderson and Fuller (1974) we have $M = A \oplus D \oplus K \oplus W$ and then $A \cong E$ is projective. On the other hand, $(W + K + D) \cap Rx \leq (W + F) \cap (Rx + F) = F + (W \cap (Rx + F)) = F + (W \cap (Ry + F)) = F + (W \cap (Ry + (W \cap F))) = F$. Hence the proof is completed. □

Corollary 2.7. *Let F be a fully invariant submodule of a module M and satisfy (S_2). Then the following conditions are equivalent.*

(1) *M is F-semiregular.*
(2) *For all $x \in M$, there exists a regular element $y \in M$ such that $x - y \in F$.*

Corollary 2.8. *Let F be a fully invariant submodule of a module M and satisfy (S_2). If $x - y \in F$ and y is F-semiregular then x is F-semiregular.*

Now we give that following lemma without proving because it can be seen by the similar proof of Nicholson (1976, Lemma 1.9).

Lemma 2.9. *Let F be a fully invariant submodule of a module M. Let $x \in M$. If $\alpha \in M^*$ is such that $(x\alpha)^2 = x\alpha$ and $x - (x\alpha)x$ is F-semiregular, then x is F-semiregular.*

By the argument in Nicholson (1976, Theorem 1.10) and Corollary 2.8, we have

Theorem 2.10. *Let F be a fully invariant submodule of a module M and $M = \bigoplus_{i\in I} M_i$ for submodules M_i. If M is F-semiregular then each M_i is F_i-semiregular where $F_i = F \cap M_i$. The converse is true if F satisfies (S_2).*

Corollary 2.11. *Let I be an ideal of a ring R with $I \leq \delta({}_R R)$. Then R is I-semiregular if and only every projective R-module M is IM-semiregular.*

Proof. Let M be a projective module. Then $IM \leq \delta(M)$ by Zhou (2000, Lemma 1.9) and so IM satisfies (S_2). Since any projective module is a summand of a free module, the proof is completed by Theorem 2.10. □

Nicholson proves the following theorem in case $F = Rad(M) \ll M$ in Nicholson (1976, Proposition 1.17). For a submodule N of M, if $N \ll_\delta M$, then N satisfies (S_1). The converse of this property is not true, for example let $M = \mathbb{Z}(p^\infty)$ be the prüfer p-group. $Rad(M) = \delta(M) = Z(M) = M$ satisfies (S_1) but not δ-small in M. Hence the following theorem generalizes Nicholson (1976, Proposition 1.17).

Theorem 2.12. *Let F be a fully invariant submodule of a module M. Consider the following conditions.*

(1) *M is F-semiregular.*
(2) (i) *Every finitely generated submodule of M/F is a direct summand.*
  (ii) *If $M/F = A/F \oplus B/F$ where A/F is finitely generated, there exists a decompsition $M = P \oplus Q$ such that $(P+F)/F = A/F$ and $(Q+F)/F = B/F$.*

Then (1) ⇒ (2)(i). *If M is projective, then* (1) ⇒ (2)(ii). *If M is projective and F satisfies* (S_1), *then* (2) ⇒ (1).

Proof. (1) ⇒ (2). Suppose M is F-semiregular and let $A/F \leq M/F$ be finitely generated. Choose a finitely generated submodule N of M such that $A/F = (N+F)/F$. By Theorem 2.3, there is a decomposition $M = C \oplus D$ such that $N = C \oplus (D \cap N)$ and $D \cap N \leq F$. Then $A/F = (N+F)/F = (C+F)/F$. Since $F = (F \cap D) \oplus (F \cap C)$ and $(D+F) \cap (C+F) = (D + (F \cap C)) \cap (C + (F \cap D)) = F$, we get $(C+F)/F \oplus (D+F)/F = M/F$. This proves (i).

Now, assume $M/F = A/F \oplus B/F$ where A/F is finitely generated. Choose N and the decomposition of M as above. Then $C + B = M$. Since C is a summand of M, apply Nicholson (1976, Lemma 1.16) to write $M = C \oplus Q$ where $Q \leq B$. Then (ii) follows because $(C+F)/F = A/F$ and $(Q+F)/F \leq B/F$.

(2) ⇒ (1). Assume that M is projective and F satisfies (S_1). Take a finitely generated submodule N of M. By (2), $M/F = (N+F)/F \oplus B/F$ for a submodule B of M with $F \leq B$. Then there exists a decomposition $M = P \oplus Q$ such that $(P+F)/F = (N+F)/F$ and $(Q+F)/F = B/F$. Hence $M = N + Q + F$. Since $F = (P \cap F) \oplus (Q \cap F), M = N + Q + (P \cap F)$. Since F satisfies (S_1), there exists a decomposition $P \cap F = K \oplus S$ where K is a summand of M and $S \ll_\delta M$. Then $M = N + Q + K + S = (N + Q + K) \oplus D$ for a submodule $D \leq S$ by Lemma 2.4. Let $T = N + Q + K$ and so T is projective. Since for a submodule L, $K \oplus L = P$ and $M = P \oplus Q = K \oplus L \oplus Q$ we get that $Q \oplus K$ is a summand of T. It gives that there is a decomposition $T = (Q \oplus K) \oplus A$ where $A \leq N$ by Nicholson (1976, Lemma 1.16). Since $(Q + K + D) \cap N \leq (Q+F) \cap (N+F) = F$, M is F-semiregular by Theorem 2.3. □

By the proof of Theorem 2.12 ($2 \Rightarrow 1$), we get the following corollary.

Corollary 2.13. *Let F be a fully invariant submodule of a module M and satisfy (S_1). If M is F-semiregular and M/F is Noetherian, then for any submodule N of M there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B \leq F$.*

3. THE SINGULAR SUBMODULE $Z(M)$

In this section, we consider the fully invariant submodule $Z(M)$ for a module M.

An R-module M is called CS (or has (C_1)), if every closed submodule is a summand. Equivalently, M is CS if and only if every submodule is essential in a summand of M. An R-module M has (C_2) if any submodule of M isomorphic to a summand of M is itself a summand. M is called *continous* if M is CS and has (C_2) (Mohamed and Müller, 1990). A module M is said to be an *ACS-module* if for every element $a \in M$, $Ra = P \oplus S$ where P is projective and S is singular (Nicholson and Yousif, 2001).

By Corollary 2.11 a ring R is left $Z({}_RR)$-semiregular if and only if every projective module M is $Z(M)$-semiregular.

If R is left $Z({}_RR)$-semiregular, then $Z({}_RR)$ satisfies (R_1) since $Z({}_RR) \leq J(R)$. Furthermore

Proposition 3.1. *Let M be a projective module with $\delta(M) \ll_\delta M$. Then the following conditions are equivalent.*

(1) *$Z(M)$ satisfies (R_1).*
(2) *$Z(M)$ satisfies (R_1).*
(3) *$Z(M) \leq \delta(M)$.*
(4) *$Z(M) \leq Rad(M)$.*

Proof. (1) $\Rightarrow$ (2). It is clear.

(2) $\Rightarrow$ (3). Since $Z(M) \cap M = Z(M), Z(M) = P \oplus S$ where P is a summand of M and $S \ll_\delta M$. Since M is projective, $P = 0$. Hence $Z(M) \ll_\delta M$.

(3) $\Rightarrow$ (4). Since $Z(M) \ll_\delta M$ and $Z(M)$ is singular, $Z(M) \ll M$.

(4) $\Rightarrow$ (1). It is clear. □

It is proved in Nicholson and Yousif (2001, Theorem 2.4) that a ring R is a left $Z({}_RR)$-semiregular if and only if R is semiregular and $J(R) = Z({}_RR)$ if and only if R is a left ACS-ring with (C_2). Now we give the module theoretic version of this result.

Theorem 3.2. *Let M be a finitely generated projective module. Then the following conditions are equivalent.*

(1) *M is $Z(M)$-semiregular.*
(2) *M is semiregular and $Z(M) = Rad(M)$.*

(3) *M is an ACS-module and every finitely generated (cyclic) projective submodule of M is a summand.*
(4) *M is an ACS-module and M has* (C_2).

Proof. $(1) \Rightarrow (2)$. If M is $Z(M)$-semiregular, then $Rad(M) \leq Z(M)$. For the converse, let $x \in Z(M)$. To show that $x \in Rad(M)$, let $L \leq M$ be such that $M = Rx + L$. Then $M/Rx \cong L/(Rx \cap L)$ is finitely generated. Let T be a finitely generated submodule of M such that $L/(Rx \cap L) = [T + (Rx \cap L)]/(Rx \cap L)$. Then $M = Rx + L = Rx + T$. By Theorem 2.3, T has a decomposition $T = P \oplus S$ where P is a projective summand of M and S is singular. Then $Rx + S \leq Z(M)$. $M = Rx + T = Rx + P + S$ and then M/P is singular. Since M is projective, $P \leq_e M$ (Nicholson and Yousif, 2001, Lemma 2.1). But this implies that $P = M$, because $P \leq^{\oplus} M$. Hence $M = T = L$. So $Rx \ll M$.

$(2) \Rightarrow (3) \Rightarrow (4)$. They are clear.

$(4) \Rightarrow (1)$. Since M is finitely generated projective, it is a summand of a finitely generated free module F. Let A be such that $F = M \oplus A$ and $\{f_i\}_{i=1}^{n}$ be a basis of F. Write $f_i = m_i + a_i$ where $m_i \in M, a_i \in A$ for all $i = 1, \ldots, n$. Let $x \in M$. By hypothesis, $Rx = P \oplus S$ where P is projective and S is singular. It is enough to show that P is a summand of M. We have an epimorphism $M \to Rx$ defined by $m = r_1 f_1 + \cdots + r_n f_n = r_1 m_1 + \cdots + r_n m_n \mapsto (r_1 + \cdots + r_n)x, m \in M, r_i \in R, 1 \leq i \leq n$. Hence, we have an epimorphism from M to P. This implies that P is isomorphic to a summand of M. By (C_2), P is a summand of M. □

It is well known that if R is left continuous then R is semiregular and $Z({}_R R) = J(R)$. By using Theorem 3.2, we prove the next result.

Theorem 3.3. *Let M be a finitely generated projective module. If M is continuous, then M is semiregular and* $Z(M) = Rad(M)$.

Proof. It is enough to show that M is an ACS-module by Theorem 3.2. Let $x \in M$. Then there exists an epimorphism $f : M \to Rx$ by the proof of $(4) \Rightarrow (1)$ of Theorem 3.2. Since M is CS, there exists a summand L of M such that $Ker(f)$ is essential in L. Let K be a submodule such that $M = L \oplus K$. Then we have isomorphisms $\alpha : Rx \to M/Ker(f)$ and $\beta : M/L \to K$. Let π denote the epimorphism from $M/Ker(f)$ to M/L. Then $g := \alpha\pi\beta : Rx \to K$ is an epimorphism. Since K is projective, g splits. There exists a homomorphism $h : K \to Rx$ such that $Rx = Im\,h \oplus Ker(g)$. $Rx/Ker(g) \cong K \cong Im\,h$ is projective and $Ker(g) = \alpha^{-1}(L/Ker(f)) \cong L/Ker(f)$ is singular. Hence Rx is a direct sum of a projective module and a singular module. □

It is well known that any finite direct sum of modules having (C_2) need not have (C_2). By Theorems 3.2 and 2.10, we have the following corollary.

Corollary 3.4. *Let M be a finitely generated projective module. If M is* $Z(M)$-*semiregular, then* $M^{(n)}$ *has* (C_2) *for every* $n \geq 1$.

The following corollary is a generalization of Yousif (1997, Proposition 1.21) and Nicholson and Yousif (2001, Corollary 2.7).

Corollary 3.5. *Let M be a finitely generated projective module. Then*

(1) *M is continuous if and only if M is $Z(M)$-semiregular and M is CS.*
(2) *The following are equivalent.*
   (a) *M is quasi-injective.*
   (b) *M is $Z(M)$-semiregular and $M \oplus M$ is CS.*
   (c) *M has (C_2) and $M \oplus M$ is CS.*
   (d) *M is continuous and $M \oplus M$ is CS.*

Proof. (1) is clear by Theorems 3.2 and 3.3. (2) (a) $\Rightarrow$ (c). By Mohamed and Müller (1990, Proposition 1.18). (c) $\Rightarrow$ (b). If $M \oplus M$ is CS, then M is CS. By Theorem 3.3, M is $Z(M)$-semiregular. (b) $\Rightarrow$ (a). By Corollary 3.4, $M \oplus M$ has (C_2). Then $M \oplus M$ is continuous. By Mohamed and Müller (1990, Theorem 3.16), M is quasi-injective. (c) $\Leftrightarrow$ (d) is clear. □

4. $\delta(M)$ AND $Soc(M)$

In this section, we investigate $\delta(M)$-semiregular and $Soc(M)$-semiregular modules. If a module M is semiregular, then it is $\delta(M)$-semiregular since $Rad(M) \leq \delta(M)$. The converse is true for finitely generated modules M with $Soc(M) = Rad(M)$ by Lemma 2.4. If M is a projective module then $\delta(M)$ is equal to the intersection of all essential maximal submodules of M (Zhou, 2000, Lemma 1.9), and hence $Soc(M) \leq \delta(M)$. So any projective $Soc(M)$-semiregular module M is $\delta(M)$-semiregular. Also we will prove in Corollary 4.6 that projective $Soc(M)$-semiregular modules are semiregular. Then we have the following implications for a projective module M.

$$M \text{ is } Soc(M)\text{-semiregular} \Longrightarrow M \text{ is semiregular} \Longrightarrow M \text{ is } \delta(M)\text{-semiregular}.$$

By Theorem 3.2, for a finitely generated projective module M, we have that

$$M \text{ is } Z(M)\text{-semiregular} \Longrightarrow M \text{ is semiregular} \Longrightarrow M \text{ is } \delta(M)\text{-semiregular}.$$

For the converse implications we give the examples at the end of the paper.

Remark 4.1. (1) Zhou (2000, Theorem 3.5), proved that R is left $\delta({}_RR)$-semiregular if and only if $R/\delta({}_RR)$ is regular and idempotents can be lifted modulo $\delta({}_RR)$. Indeed this result follows from Theorem 2.12 because $\delta({}_RR)$ satisfies (S_2).

(2) Also $Soc({}_RR)$ satisfies (S_2), since $Soc({}_RR) \leq \delta({}_RR)$. Hence R is left $Soc({}_RR)$-semiregular if and only if $R/Soc({}_RR)$ is regular and idempotents can be lifted modulo $Soc({}_RR)$. Baccella proved that for any ring R, idempotents can be lifted modulo $Soc({}_RR)$ (see Yousif and Zhou, 2002, Lemma 1.2). Thus R is left $Soc({}_RR)$-semiregular if and only if $R/Soc({}_RR)$ is regular (see Yousif and Zhou, 2002, Theorem 1.6).

By Corollary 2.11, a ring R is left $Soc(_R R)(\delta(_R R))$-semiregular if and only if every projective module M is $Soc(M)(\delta(M))$-semiregular.

The next result is a structure theorem for countably generated $\delta(\cdot)$-semiregular modules.

Theorem 4.2. *Let M be a countably generated $\delta(M)$-semiregular module. If $\delta(M)$ is δ-small in M then M is isomorphic to a direct sum of projective cyclic submodules.*

Proof. Let $x_1, x_2, \ldots$ be a generating set for M. There is a decomposition $M = P_1 \oplus Q_1$ such that $P_1 \leq Rx_1$ is projective and $K_1 = Q_1 \cap Rx_1$ is δ-small in M. As a summand of Rx_1, the module P_1 is cyclic. Now we use induction. Assume, for a positive integer n, M has a decomposition $M = (\sum_{i=1}^{n} P_i) \oplus Q_n$ such that $\sum_{i=1}^{n} Rx_i \subset (\bigoplus_{i=1}^{n} P_i) + K_n$, where K_n is δ-small in M.

Since Q_n is a summand of M and $\delta(Q_n) = Q_n \cap \delta(M)$, Q_n is $\delta(Q_n)$-semiregular. Then there is a decomposition $Q_n = P_{n+1} \oplus Q_{n+1}$ such that $P_{n+1} \leq Rx_{n+1}$ is projective and $T = Q_{n+1} \cap Rx_{n+1}$ is δ-small in Q_n. Hence $M = (\sum_{i=1}^{n+1} P_i) \oplus Q_{n+1}$ and $\sum_{i=1}^{n+1} Rx_i \subset (\bigoplus_{i=1}^{n+1} P_i) + K_{n+1}$, where $K_{n+1} = K_n + T$ is δ-small in M. Since $K = \sum_{i\in\mathbb{N}} K_i \leq \delta(M)$, it is δ-small in M and by Lemma 2.4 there exists a projective semisimple submodule P of K such that $M = \sum_{i\in\mathbb{N}} Rx_i = (\bigoplus_{i\in\mathbb{N}} P_i) + K = (\bigoplus_{i\in\mathbb{N}} P_i) \oplus P$. The proof is completed. □

Corollary 4.3. *Any finitely generated $\delta(M)$-semiregular module M is projective and $Z(M) \leq Rad(M)$.*

Proof. By Theorem 2.3 and Proposition 3.1, $Z(M) \leq Rad(M)$. □

Since every projective module is a direct sum of countably generated submodules we have,

Corollary 4.4. *Any projective $\delta(M)$-semiregular module M with $\delta(M) \ll_\delta M$ is isomorphic to a direct sum of cyclic submodules.*

We have mentioned that if M is a projective $Soc(M)$-semiregular module then M is $\delta(M)$-semiregular. These modules are also semiregular and hence this result is a generalization of Yousif and Zhou (2002, Corollary 1.7(2)).

Theorem 4.5. *If M is a $Soc(M)$-semiregular module and $Z(M) \leq Rad(M)$, then M is semiregular.*

Proof. Let $x \in M$ and $M = A \oplus B$ where $A \leq Rx$ is projective and $Rx \cap B \leq Soc(M)$. Then $Rx = A \oplus (Rx \cap B)$. Assume that $Rx \cap B$ has a simple submodule S_1 such that $S_1 \not\subseteq Rad(M)$, if not every simple submodule of $Rx \cap B$ is in $Rad(M)$ and hence this completes the proof. Then S_1 is a summand of M, and hence summand of B. Let L_1 be such that $B = S_1 \oplus L_1$. Then $Rx \cap B = S_1 \oplus (Rx \cap L_1)$ and $M = A \oplus S_1 \oplus L_1$. This implies that $Rx = (A \oplus S_1) \oplus (Rx \cap L_1)$.

Similarly since $Rx \cap L_1$ is semisimple assume that $Rx \cap L_1$ has a simple submodule S_2 such that $S_2 \not\subseteq Rad(M)$, if not again the proof is completed. Since S_2 is a summand of M, there exists a submodule L_2 such that $L_1 = S_2 \oplus L_2$. It follows that $Rx \cap L_1 = S_2 \oplus (Rx \cap L_2)$ and $M = A \oplus S_1 \oplus S_2 \oplus L_2$. Then $Rx = (A \oplus S_1 \oplus S_2) \oplus (L_2 \cap Rx)$. This process produces a strictly descending chain $B \cap Rx \supset L_1 \cap Rx \supset L_2 \cap Rx \supset \cdots$. Since $B \cap Rx$ is semisimple and finitely generated, it is Artinian. Hence this process must stop, so that $L_n \cap Rx \leq Rad(M)$ for some positive integer n. Hence $Rx = (A \oplus S_1 \oplus \cdots \oplus S_n) \oplus (L_n \cap Rx)$. So M is semiregular. □

Corollary 4.6. *Any projective $Soc(M)$-semiregular module M is semiregular.*

Proof. Since $Z(M) \leq Soc(M)$, let S be a singular simple submodule of M. If $S \not\subseteq Rad(M)$, then S is a summand of M. This implies that $S = 0$. Hence $Z(M) \leq Rad(M)$. By Theorem 4.5, M is semiregular. □

Corollary 4.7. *Let M be a finitely generated $Soc(M)$-semiregular module. Then M is projective if and only if $Z(M) \leq Rad(M)$.*

Proof. It is clear by Theorem 4.5 and Corollary 4.3. □

Hence if M is a projective $Soc(M)$-semiregular module then

$$Z(M) \leq Rad(M) \leq Soc(M) \leq \delta(M).$$

If R is a left $Soc({}_RR)$-semiregular ring, then $\delta({}_RR) = Soc({}_RR)$. For, $\delta({}_RR)/Soc({}_RR) = J(R/Soc({}_RR)) = 0$ (Zhou, 2000, Corollary 1.7). Also $J(R)^2 = 0$ because $J(R)Soc({}_RR) = 0$. But this does not necessarily hold if R is semiregular. For example there exists a local ring R such that $J(R)$ is not nilpotent (see Zhou, 2000, Example 4.4 for the existence of such a ring). Then R is semiregular but $J(R)^2 \neq 0$.

Proposition 4.8. *If a module M is $Soc(M)$-semiregular, then M is an ACS-module.*

Proof. Let $a \in M$. Then $Ra = A \oplus B$ where A is a projective summand of M and $B \leq Soc(M)$. Let $B = B_1 \oplus B_2$ where B_1 is a direct sum of projective simples and B_2 is a direct sum of singular simples. Then $Ra = A \oplus B_1 \oplus B_2$ where $A \oplus B_1$ is projective and B_2 is singular. □

Next we consider the Noetherian $Soc(M)$-semiregular modules.

Theorem 4.9. *Any Noetherian $Soc(M)$-semiregular module M is Artinian.*

Proof. If M is Noetherian $Soc(M)$-semiregular, $M/Soc(M)$ is semisimple by Theorem 2.12. Since M is Noetherian, $M/Soc(M)$ is Artinian and so M is Artinian. □

Corollary 4.10. *The following conditions are equivalent for a ring R.*

(1) *R is a left Artinian ring with $J(R)^2 = 0$.*
(2) *R is a left Noetherian left $Soc(_RR)$-semiregular ring.*

Proof. (2) ⇒ (1). It is clear.

(1) ⇒ (2). Since the left annihilator of $J(R)$ is $Soc(_RR)$, $J(R) \leq Soc(_RR)$. Left Artinian rings are semiregular. Hence R is left $Soc(_RR)$-semiregular. □

From now on, we deal with $Soc(M)$-semiregular modules M such that M has (C_2) or is min-CS or CS.

Proposition 4.11. *Let M be a finitely generated projective module. Then the following conditions are equivalent.*

(1) *M is $Soc(M)$-semiregular with (C_2).*
(2) *M is $Soc(M)$-semiregular and $Z(M) = Rad(M)$.*
(3) *M is $Soc(M)$-semiregular and every simple projective submodule of M is a summand.*
(4) *M is $Z(M)$-semiregular and $Z(M) \leq Soc(M)$.*

Proof. (4) ⇒ (1) and (4) ⇒ (2) are clear. (1) ⇒ (4) is by Theorem 3.2 and Proposition 4.8

(2) ⇒ (3). Let S be a projective simple submodule of M. Then $S \nsubseteq Rad(M)$ and hence S is a summand of M.

(3) ⇒ (4). Let $x \in M$. Then M has a decompositon $M = A \oplus B$ such that A is a projective submodule of Rx and $B \cap Rx \leq Soc(M)$. Then $Rx = A \oplus (B \cap Rx)$. Let $B \cap Rx = S_1 \oplus S_2$ where S_1 is a finite direct sum of projective simples and S_2 is a finite direct sum of singular simples. Then S_1 is a summand of B by the similar proof of Mohamed and Müller (1990, Proposition 2.2). Hence $A \oplus S_1$ is a summand of M. This implies that M is $Z(M)$-semiregular. □

By Theorems 2.10 and 3.2, if M is a finitely generated projective $Soc(M)$-semiregular module with (C_2), then $M^{(n)}$ is $Soc(M^{(n)})$-semiregular and has (C_2) for every $n \geq 1$.

For the following corollary see also Yousif and Zhou (2002, Theorem 2.11).

Corollary 4.12. *The following conditions are equivalent for a ring R.*

(1) *R is left $Soc(_RR)$-semiregular, $R/Soc(_RR)$ is Noetherian and any projective semisimple left ideal is a summand.*
(2) *R is semiprimary and $J(R) = Z(_RR) \leq Soc(_RR)$.*

Proof. (1) ⇒ (2). By Corollary 2.13 and the hypothesis, R is semiperfect. Since $J(R)^2 = 0$, R is semiprimary. By Proposition 4.11, $J(R) = Z(_RR)$.

(2) ⇒ (1). Since R is semiprimary, it is semiregular and R/J(R) is semisimple Artinian. Since $J(R) \leq Soc(_R R)$, R is left $Soc(_R R)$-semiregular and $R/Soc(_R R)$ is Noetherian. Since $J(R) = Z(_R R)$, any projective semisimple left ideal is a summand. □

A module M is called a *min-CS module* if every simple submodule of M is essential in a summand of M. *A* ring R is called left *min-CS* ring if $_R R$ is a min-CS module.

Proposition 4.13. *Let M be a Noetherian projective module. Then the following conditions are equivalent.*

(1) *M is continuous and* $Rad(M) \leq Soc(M)$.
(2) *M is* $Soc(M)$*-semiregualr, min-CS with* (C_2).

Proof. (1) ⇒ (2). It is clear by Theorem 3.3.

(2) ⇒ (1). We claim that M is CS. Let N be a submodule of M. Then N has a decomposition $N = A \oplus S$ such that A is a summand of M and $S \leq Soc(M)$. Since M is min-CS and by Mohamed and Müller (1990, Proposition 2.2), there exists a summand C of M such that $S \leq_e C$. Then $N \leq_e A \oplus C \leq^{\oplus} M$. Hence M is CS. □

A ring R is called *left Kasch* if every simple left R-module is embedded in R, or equivalently, for any maximal left ideal I in R, the right annihilator of I is nonzero. By Therorem 4.9 and Yousif (1997, Theorem 1.16), we have the following corollary.

Corollary 4.14. *Let R be a left Noetherian ring. The following conditions are equivalent.*

(1) *R is left continuous with* $J(R) \leq Soc(_R R)$.
(2) *R is left* $Soc(_R R)$*-semiregular left min-CS and left* (C_2).

In this case R is a left Artinian left and right Kasch ring.

If a ring R is left Artinian left continuous left and right Kasch with $J(R) \leq Soc(_R R)$, R need not be a QF-ring:

Example 4.15 (Björk, 1970). Given a field F and an isomorphism a $\mapsto \overline{a}$ from $F \to \overline{F} \subseteq F$, let R be the right F-space on basis {1, t} with multiplication given by $t^2 = 0$ and at = t $\overline{a}$ for all a $\in F$. Then R is a local ring and the only right ideals are 0, $J(R)$ and R. Hence R is right Artinian right continuous and left and right Kasch. It follows that $J(R) = Soc(_R R) = Soc(_R R)$. If $\dim_{\overline{F}}(F) \geq 2$, then R is not left continuous (see Yousif and Zhou, 2002, Example 2.17).

Theorem 4.16. *Let M be a finitely generated module. Then the following conditions are equivalent.*

(1) *M is CS and* $M/Soc(M)$ *is semisimple.*
(2) *M is CS Artinian and* $Rad(M) \leq Soc(M)$.

In addition if M is projective, (1) *and* (2) *are equivalent to*

(3) *M is CS $Soc(M)$-semiregular and $M/Soc(M)$ is Noetherian.*

Proof. (1) $\Rightarrow$ (2). Since $M/Soc(M)$ is semisimple, $Rad(M) \leq Soc(M)$. By Dung et al. (1994, 5.15 and 18.7), M is Artinian.

(2) $\Rightarrow$ (1). Since M is Artinian, $M/Rad(M)$ is semisimple.

(2) $\Rightarrow$ (3). Since M is Artinian and projective, M is semiregular (Wisbauer, 1991, 41.15) and $M/Rad(M)$ is semisimple. Then M is $Soc(M)$-semiregular and $M/Soc(M)$ is semisimple.

(3) $\Rightarrow$ (1). By Theorem 2.12, $M/Soc(M)$ is semisimple. □

Corollary 4.17. *The following conditions are equivalent for a ring R.*

(1) *R is left CS left Artinian with $J(R)^2 = 0$.*
(2) *R is left CS left $Soc(_RR)$-semiregular and $R/Soc(_RR)$ is left Noetherian.*

Theorem 4.18. *Let M be finitely generated projective module. The following conditions are equivalent.*

(1) *M is Artinian quasi-injective and $Rad(M) \leq Soc(M)$.*
(2) *M has (C_2), $M \oplus M$ is CS and $M/Soc(M)$ is semisimple.*
(3) *M is Noetherian $Soc(M)$-semiregular with (C_2) and $M \oplus M$ is min-CS.*

Proof. (1) $\Rightarrow$ (2). Since M is quasi-injective, $M \oplus M$ is *CS* by Mohamed and Müller (1990, Proposition 1.18).

(2) $\Rightarrow$ (3). Since M is *CS* and $M/Soc(M)$ is Artinian and Noetherian, M is Artinian and Noetherian by Dung et al. (1994, 5.15 and 18.17). Since M is Artinian and projective, it is semiregular (Wisbauer, 1991, 41.15). Since $Rad(M) \leq Soc(M)$, M is $Soc(M)$-semiregular.

(3) $\Rightarrow$ (1). Then $M \oplus M$ is $Soc(M \oplus M)$-semiregular and by Proposition 4.11 and 4.13, $Z(M \oplus M) = Rad(M \oplus M)$ and $M \oplus M$ is continuous. Hence M is quasi-injective (Mohamed and Müller, 1990, Theorem 3.16). □

Note that a left self-injective (resp. right and left continuous) ring R such that $R/Soc(_RR)$ is left Noetherian is QF (Ara and Park, 1991). But there exists a Noetherian projective self-injective module which is not Artinian (see Dung et al., 1994, Example in p. 87). Hence in the above theorem it is not enough for M to be Artinian to assume that $M/Soc(M)$ is Noetherian.

Corollary 4.19. *The following conditions are equivalent for a ring R.*

(1) *R is a QF-ring with $J(R)^2 = 0$.*
(2) *$_RR$ has (C_2), $_R(R \oplus R)$ is CS and $R/Soc(_RR)$ is semisimple Artinian.*
(3) *R is left $Soc(_RR)$-semiregular, left Noetherian with left (C_2) and $R \oplus R$ is left min-CS.*

Now we give the examples. First example shows that there is a projective module M which is $\delta(M)$-semiregular but not semiregular hence not $Soc(M)$-semiregular (see Nicholson, 1976, Example 2.15).

Example 4.20. Let F be a field, $I = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$ and

$$M = R = \{(x_1, \dots, x_n, x, x, \dots) : n \in \mathbb{N}, x_i \in M_2(F), x \in I\}.$$

With component-wise operations, R is a ring.

$$\delta(_R R) = \{(x_1 \dots, x_n, x, x, \dots) : n \in \mathbb{N}, x_i \in M_2(F), x \in J\} \quad \text{where} \quad J = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}.$$

$$Soc(_R R) = \{(x_1, \dots, x_n, 0, 0, \dots) : n \in \mathbb{N}, x_i \in M_2(F)\}$$

Thus,

$$R/Soc(_R R) \cong \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$$

and so R is not left $Soc(_R R)$-semiregular. Also by Example 2.15 in Nicholson (1976) R is not semiregular, but $\delta(_R R)$-semiregular by Example 4.3 in Zhou (2000).

If M is finitely generated projective $Z(M)$-semiregular, then M need not be $Soc(M)$-semiregular. Hence there is a module M which is semiregular but not $Soc(M)$-semiregular (see also Yousif and Zhou, 2002, Example 1.8).

Example 4.21. Let $M = R = \mathbb{Z}_8$. Then R is a self-injective ring, $J(R) = Z(R) = 2R$ and $Soc(R) = 4R$. Hence R is a $Z(R)$-semiregular ring by Nicholson and Yousif (2001) but not $Soc(R)$-semiregular since $J(R)$-semiregular is not contained in $Soc(R)$.

If M is $Soc(M)$-semiregular then M need not be $Z(M)$-semiregular. The ring of 2×2 upper triangular matrices over a field is the example of such a module, see Yousif and Zhou, 2002, Example 1.8).

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