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Hybrid Numbers and Roots of Matrices

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Abstract. In this study, we give a new method for finding n-th roots of a 2×2 real matrix with the help of hybrid numbers. We define argument and polar forms of a 2×2 matrix and express the De Moivre's formulas according to the type and character of the matrix.

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1. Introduction

A matrix B is said to be an n-th root of a matrix A if $B^n = A$, where $n \geq 2$. There are many studies in the literature giving different methods of finding roots of a matrix. These methods mainly depend on the Shur Theorem, the Cayley Hamilton Theorem or the Newton Method. Denman described an algorithm for computing roots of a real matrix with the real part of eigenvalues not zero in 1981 [5]. After that, Björk and Hammarling developed a method for calculating the square root of a matrix based upon the Schur factorization method [2]. Higham described a generalization of the Schur factorization method for the real square root of a n by n matrix in 1987 [6]. The n-th root of a matrix A may not exist. In this case, A is called rootless matrix. Some of the known results related to the existence of a root of a matrix are also as follows. If a matrix A is nonsingular and diagonalizable then A always has a root. Yuttanan also examines roots of nilpotent matrices in [19]. On the other hand, if an $n \times n$ matrix has at least $n-1$ nonzero eigenvalues, then this matrix has a square root.

In this study, we will not be interested in the roots of $n \times n$ matrices. We will especially deal with the finding of the n-th roots of 2×2 matrices. Sullivan described a method to calculate all square roots of a 2×2 matrix, using the Cayley Hamilton theorem [17]. The Newton method used to find roots of a 2x2 matrix can be found in the Higham's [7] and Lannazzo's [9] papers. Some of the basic methods to find square roots of a 2×2 matrix are summarized in the Nortshield's paper [13]. Moreover, some of the recent studies related to the finding of the roots of 2×2 matrices are as follows [1], [3], [8], [10], [13],

[14], [15], [16], [17]. In 2004, Choudhry was concerned with the determination of an algebraic formula giving the nth roots of 2×2 matrices. If A is a 2×2 scalar matrix, the equation $B^n = A$ has infinitely many solutions, if A is a non-scalar 2×2 matrix, the equation $B^n = A$ has a finite number of solutions and Choudhry gives a formula expressing all solutions in terms of A [3].

In this study, we will provide a new algebraic method that is different from the above methods. For this, we will take advantage of the concept of the hybrid number defined in the previous article of Özdemir [11]. With the help of hybrid numbers, we will define a polar form of any real matrix and give the De Moivre formula for 2×2 matrices to find the roots of a matrix. The paper is organized in the following way: in section 2, we give the definition of hybrid numbers and some important Theorems. Also, we classify 2×2 matrices by using hybrid numbers. For detailed information on hybrid numbers see [11], [4]. In Section 3, we describe the polar representation of a 2×2 matrix. In section 4, de Moivre's formulas for 2×2 matrices are proved and methods of finding the roots of a 2×2 matrix are examined separately according to its character and type.

2. Hybrid Numbers and Classification of 2×2 Matrices

Hybrid numbers are a new generalization of complex, hyperbolic and dual numbers. It is a noncommutative ring. We can classify a Hybrid number as elliptic, hyperbolic, or parabolic. On the other hand, a hybrid number is classified as timelike, spacelike and lightlike according to its norm. As in the complex numbers, we can find roots of a hybrid number, using the De Moivre's formula for hybrid numbers. There is an isomorphism between the algebra of hybrid numbers and algebra of 2×2 real matrices. So, it can be classified 2×2 matrices, with respect to kind of corresponding hybrid number. Using this isomorphism, we will define De Moivre formula for 2x2 matrices and find $n-$roots of a 2×2 matrix.

Definition 2.1. The set of hybrid numbers $\mathbb{K}$, defined as

$$\mathbb{K} = \left\{ a+b\mathbf{i}+c\varepsilon+d\mathbf{h} : a,b,c,d \in \mathbb{R},\ \mathbf{i}^2 = -1,\ \varepsilon^2 = 0,\ \mathbf{h}^2 = 1,\ \mathbf{ih} = -\mathbf{hi} = \varepsilon + \mathbf{i} \right\}.$$

For the hybrid number $\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$, the number a is called the scalar part and is denoted by $S(\mathbf{Z})$. The part $b\mathbf{i} + c\varepsilon + d\mathbf{h}$ is also called the vector part and is denoted by $V(\mathbf{Z})$. Multiplication table of hybrid numbers as follows.

$\cdot$	$\boldsymbol{i}$	$\boldsymbol{\varepsilon}$	$\boldsymbol{h}$
$\boldsymbol{i}$	-1	$\boldsymbol{1-h}$	$\boldsymbol{\varepsilon+i}$
$\boldsymbol{\varepsilon}$	$\boldsymbol{h+1}$	0	$-\varepsilon$
$\boldsymbol{h}$	$\boldsymbol{-\varepsilon-i}$	ε	1

Multiplication operation in the hybrid numbers is associative and not commutative. The conjugate of a hybrid number $\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$, denoted by $\overline{\mathbf{Z}}$, is defined as $\overline{\mathbf{Z}} = S(\mathbf{Z}) - V(\mathbf{Z}) = a - b\mathbf{i} - c\varepsilon - d\mathbf{h}$ as in quaternions.

Definition 2.2. **(Character of a Hybrid Number)** Let $\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ be a hybrid number. The real number

$$\mathcal{C}(\mathbf{Z}) = \mathbf{Z}\overline{\mathbf{Z}} = \overline{\mathbf{Z}}\mathbf{Z} = a^2 + (b-c)^2 - c^2 - d^2 \tag{2.1}$$

is called the characteristic number of $\mathbf{Z}$. We say that a hybrid number;

$$\begin{cases} \mathbf{Z} \text{ is spacelike} & \text{if } \mathcal{C}(\mathbf{Z}) < 0; \\ \mathbf{Z} \text{ is timelike} & \text{if } \mathcal{C}(\mathbf{Z}) > 0; \\ \mathbf{Z} \text{ is lightlike} & \text{if } \mathcal{C}(\mathbf{Z}) = 0. \end{cases}$$

These are called the **characters of the hybrid numbers**.

Definition 2.3. **(Type of a Hybrid Number)** Let $\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ be a hybrid number. The real number

$$\triangle(\mathbf{Z}) = -(b-c)^2 + c^2 + d^2$$

is called the type number of $\mathbf{Z}$. We say that a hybrid number;

$$\begin{cases} \mathbf{Z} \text{ is elliptic} & \text{if } \triangle(\mathbf{Z}) < 0; \\ \mathbf{Z} \text{ is hyperbolic} & \text{if } \triangle(\mathbf{Z}) > 0; \\ \mathbf{Z} \text{ is parabolic} & \text{if } \triangle(\mathbf{Z}) = 0. \end{cases}$$

These are called the **types of the hybrid numbers**. Also, the vector $\mathcal{E}_{\mathbf{Z}} = (b-c, c, d)$ is called hybridian vector of $\mathbf{Z}$.

	THE CHARACTER		
	Spacelike	Lightlike	Timelike
TYPE	Hyperbolic	Hyperbolic	Hyperbolic
		Parabolic	Parabolic
			Elliptic

Definition 2.4. **(Norms of Hybrid Numbers)** Let $\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ be a hybrid number. The real number

$$\|\mathbf{Z}\| = \sqrt{|\mathcal{C}(\mathbf{Z})|} = \sqrt{\left|a^2 + (b-c)^2 - c^2 - d^2\right|}$$

is called **norm of Z**. Besides, the real number

$$\mathcal{N}(\mathbf{Z}) = \sqrt{|\triangle|} = \sqrt{\left|-(b-c)^2 + c^2 + d^2\right|}$$

will be called the norm of the hybrid vector of $\mathbf{Z}$.

Remark 2.5. This norm definition is a generalized norm definition that overlaps with the definitions of norms in complex, hyperbolic and dual numbers. Actually,

1. If Z is a complex number $(c = d = 0)$, then $\|\mathbf{Z}\| = \sqrt{|\mathbf{Z}\overline{\mathbf{Z}}|} = \sqrt{a^2 + b^2}$,
2. If Z is a hyperbolic number $(b = c = 0)$, then $\|\mathbf{Z}\| = \sqrt{|a^2 - d^2|}$,
3. If Z is a dual number $(b = d = 0)$, then $\|\mathbf{Z}\| = \sqrt{a^2} = |a|$.

Using the hybridian product of hybrid numbers, one can show that the equality $\mathcal{C}(\mathbf{Z}_1\mathbf{Z}_2)=\mathcal{C}(\mathbf{Z}_1)\,\mathcal{C}(\mathbf{Z}_2)$. So, timelike hybrid numbers form a group according to the multiplication operation. The inverse of the number of the hybrid number $\mathbf{Z}=a+b\mathbf{i}+c\varepsilon+d\mathbf{h}$, $\|\mathbf{Z}\|\neq 0$ is defined as

$$\mathbf{Z}^{-1}=\frac{\overline{\mathbf{Z}}}{\mathcal{C}(\mathbf{Z})}.$$

Accordingly, lightlike hybrid numbers have no inverse.

Definition 2.6. Let $\mathbf{Z}=a+b\mathbf{i}+c\varepsilon+d\mathbf{h}$ be a hybrid number. Argument of $\mathbf{Z}$ is defined as follows with respect to its type.

$$\arg\mathbf{Z}=\theta=\begin{cases}\pi-\arctan\dfrac{\mathcal{N}(\mathbf{Z})}{a} & \text{If } \mathbf{Z} \text{ is elliptic and } a<0,\\ \arctan\dfrac{\mathcal{N}(\mathbf{Z})}{a} & \text{If } \mathbf{Z} \text{ is elliptic and } a>0,\\ \ln\left|\dfrac{a+\mathcal{N}(\mathbf{Z})}{\rho}\right| & \text{If } \mathbf{Z} \text{ is nonlightlike hyperbolic;}\\ \dfrac{c}{\|\mathbf{Z}\|} & \text{If } \mathbf{Z} \text{ is parabolic.}\end{cases}\tag{2.2}$$

Theorem 2.7. *Let $\mathbf{Z}=a+b\mathbf{i}+c\varepsilon+d\mathbf{h}$ be a hybrid number, and $\theta=\arg\mathbf{Z}$.*

i. If $\mathbf{Z}$ is elliptic, then $\mathbf{Z}=\rho(\cos\theta+\mathbf{U}\sin\theta)$ such that $\mathbf{U}^2=-1$;

ii. If $\mathbf{Z}$ a lightlike hyperbolic, then $\mathbf{Z}=a(1+\mathbf{U})$ such that $\mathbf{U}^2=1$;

iii. If $\mathbf{Z}$ is spacelike or timelike hyperbolic, then, $\mathbf{Z}=k\rho(\cosh\theta+\mathbf{U}\sinh\theta)$ such that $\mathbf{U}^2=1$, where $\rho=\|\mathbf{Z}\|$, $\mathbf{U}=\frac{b\mathbf{i}+c\varepsilon+d\mathbf{h}}{\mathcal{N}(\mathbf{Z})}$ and

$$k=\begin{cases}1 & \mathbf{Z} \text{ is timelike and } a>0,\\ -1 & \mathbf{Z} \text{ is timelike and } a<0,\\ \mathbf{U} & \mathbf{Z} \text{ is spacelike and } a>0,\\ -\mathbf{U} & \mathbf{Z} \text{ is spacelike and } a<0,\end{cases}$$

for $k\in\{-1,1,\mathbf{U},-\mathbf{U}\}$

iv. If $\mathbf{Z}$ is a parabolic hybrid number, then $\mathbf{Z}=\|\mathbf{Z}\|(\xi+\mathbf{U})$ where $\mathbf{U}=\frac{V(\mathbf{Z})}{\rho}$, $\mathbf{U}^2=0$, $\xi=sgn(S(\mathbf{Z}))$.

Theorem 2.8. *(Özdemir 2018) Let $\mathbf{Z}=a+\mathbf{U}b$, $\mathbf{U}^2\in\{\pm1,0\}$ be a spacelike or timelike hybrid number. If $\theta=\arg\mathbf{Z}$ and $\rho=\|\mathbf{Z}\|$.*

i. If $\mathbf{Z}$ is elliptic, then $\mathbf{Z}^n=\rho^n(\cos n\theta+\mathbf{U}\sin n\theta)$, $\mathbf{U}^2=-1$;

ii. If $\mathbf{Z}$ is hyperbolic, then $\mathbf{Z}^n=k^n\rho^n(\cosh n\theta+\mathbf{U}\sinh n\theta)$, $\mathbf{U}^2=1$;

iii. If $\mathbf{Z}$ is parabolic, then $\mathbf{Z}^n=\rho^n\left(\xi^n+n\xi^{n-1}\mathbf{U}\right)$, $\mathbf{U}^2=0$.

Theorem 2.9. *(Özdemir 2018)If $\mathbf{Z}=a(1+\mathbf{U})$ is a lightlike hybrid number, then $\mathbf{Z}^n=a^n2^{n-1}(1+\mathbf{U})$ where $\mathbf{U}=\frac{V(\mathbf{Z})}{\mathcal{N}(\mathbf{Z})}$ and $\mathbf{U}^2=1$.*

Let $\mathbf{W}\in\mathbb{K}$ and $n\in\mathbb{Z}^+$, the hybrid numbers $\mathbf{Z}$ satisfying the equation $\mathbf{Z}^n=\mathbf{W}$ is called the root of the n-th degree of the hybrid number $\mathbf{W}$. Finding the root of a hybrid number will vary depending on the type (parabolic, elliptic, hyperbolic) and character (timelike, spacelike, lightlike) of this hybrid number. The roots of a hybrid number can be given as follows in two separate cases as in the split quaternions [12]. The first case is for spacelike or timelike hybrid numbers, the second case is for lightlike hybrid numbers.

Theorem 2.10. *(Özdemir 2018) Let* $\mathbf{W}$ *be a hybrid number and* $n \in \mathbb{Z}^+$. *The hybrid numbers* $\mathbf{Z}$ *satisfying the equation* $\mathbf{Z}^n = \mathbf{W}$ *can be found as follows.*

i. If $\mathbf{W} = \rho(\cos\theta + \mathbf{U}\sin\theta)$ *is an elliptic hybrid number, then the roots of* $\mathbf{W}$ *are in the form*

$$\mathbf{Z}_m = \sqrt[n]{\rho}\left(\cos\frac{\theta + 2m\pi}{n} + \mathbf{U}\sin\frac{\theta + 2m\pi}{n}\right) \tag{2.3}$$

for $m = 0, 1, 2, ..., n-1$;

ii. If $\mathbf{W} = \rho k(\cosh\theta + \mathbf{U}\sinh\theta)$ *is a spacelike or timelike hyperbolic hybrid number, then the roots of* $\mathbf{W}$ *are in the form*

$$\mathbf{Z}_k = \begin{cases} \sqrt[n]{\rho}\left(\cosh\dfrac{\theta}{n} + \mathbf{U}\sinh\dfrac{\theta}{n}\right) & \textit{If } n \textit{ is odd,} \\ k\sqrt[n]{\rho}\left(\cosh\dfrac{\theta}{n} + \mathbf{U}\sinh\dfrac{\theta}{n}\right) & \textit{If } n \textit{ is even, } \mathbf{W} \textit{ is timelike and } a > 0, \\ \textit{No roots} & \textit{other cases} \end{cases}$$

where $k \in \{1, -1, \mathbf{U}, -\mathbf{U}\}$;

iii. If $\mathbf{W} = \rho(\xi + \mathbf{U})$, $\xi = sgn(S(\mathbf{Z}))$ *is a parabolic hybrid number, the only root is*

$$\mathbf{Z} = \rho\left(1 + \frac{\mathbf{U}}{n}\right)$$

where $\rho = \|\mathbf{Z}\|$.

Theorem 2.11. *(Özdemir 2018) If* $\mathbf{W} = a(1 + \mathbf{U})$ *is a lightlike hybrid number, then*

$$\mathbf{Z} = \begin{cases} \dfrac{\pm\sqrt[n]{2a}}{2}(1 + \mathbf{U}) & \textit{if } n \textit{ is even} \\ \dfrac{\sqrt[n]{2a}}{2}(1 + \mathbf{U}) & \textit{if } n \textit{ is odd} \end{cases}$$

for $n \in \mathbb{Z}^+$ *where* $\mathbf{U} = \frac{V(\mathbf{Z})}{\mathcal{N}(\mathbf{Z})}$ *and* $\mathbf{U}^2 = 1$.

2.1. Classification of 2×2 Matrices Using Hybrid Numbers

Just as we classify a hybrid numbers, we can classify a 2×2 matrix. Any 2×2 matrix is classified as spacelike, timelike or lightlike and sorted as hyperbolic, elliptic, or parabolic, taking into account the isomorphisms and relations between hybrid numbers and 2×2 matrices.

Theorem 2.12. *(Özdemir 2018) The ring of hybrid numbers* $\mathbb{K}$ *is isomorphic to the ring of real 2×2 matrices* $\mathbb{M}_{2\times 2}$ *with the map* $\varphi : \mathbb{K} \to \mathbb{M}_{2\times 2}$ *where*

$$\varphi(a + b\mathbf{i} + c\varepsilon + d\mathbf{h}) = \begin{bmatrix} a + c & b - c + d \\ c - b + d & a - c \end{bmatrix} \tag{2.4}$$

for $\mathbf{Z} = a + b\mathbf{i} + c\varepsilon + d\mathbf{h} \in \mathbb{K}$.

The matrix $\varphi(\mathbf{Z}) \in \mathbb{M}_{2\times 2}(\mathbb{R})$ is called the hybrid matrix corresponding to the hybrid number $\mathbf{Z}$. By defining this isomorphism between 2×2 matrices and hybrid numbers, we can easily multiply the hybrid numbers and prove many features of hybrid numbers more easily. Also, we have

$$\varphi^{-1}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \left(\frac{a+d}{2}\right) + \left(\frac{a+b-c-d}{2}\right)\mathbf{i} + \left(\frac{a-d}{2}\right)\varepsilon + \left(\frac{b+c}{2}\right)\mathbf{h}. \quad (2.5)$$

Theorem 2.13. *(Özdemir 2018) Let A be a 2×2 real matrix corresponding to the hybrid number $\mathbf{Z}$, then there are the following relations.*

i. $\rho = \|\mathbf{Z}\| = \sqrt{|\det A|}$,

ii. $\triangle(\mathbf{Z}) = \left(\frac{\mathrm{tr}A}{2}\right)^2 - \det A$,

iii. $P(\lambda) = \lambda^2 - (\mathrm{tr}A)\lambda + \det A$, $\triangle_A = (\mathrm{tr}A)^2 - 4\det A = 4\triangle(\mathbf{Z})$ *is discriminant of the characteristic polynomial of* A.

iv. $\mathbf{Z}^{-1}$ *exists if and only if* $\det(A) \neq 0$.

As a conclusion of this Theorem, we had shown that the classification of hybrid numbers depends entirely on the determinant and the trace of the 2×2 corresponding matrices. That is, we can classify a hybrid number $\mathbf{Z}$, with respect to the kind of the corresponding matrix $\varphi(\mathbf{Z}) = A$. Besides, we can classify 2×2 matrices, similar to hybrid numbers. Thus, we can give the following classifications for 2×2 real matrices.

Definition 2.14. Let A be a 2×2 real matrix. Then,

$$\begin{cases} \mathbf{Z} \text{ is spacelike} & \text{if } \det A < 0; \\ \mathbf{Z} \text{ is timelike} & \text{if } \det A > 0; \\ \mathbf{Z} \text{ is lightlike} & \text{if } \det A = 0. \end{cases} \quad (2.6)$$

Definition 2.15. Let A be a 2×2 real matrix where λ_1 and λ_2 are the eigenvalues of A. Then,

$$\begin{cases} A \text{ is called elliptic} & \text{if } \lambda_1, \lambda_2 \text{ are complex numbers;} \\ A \text{ is called hyperbolic} & \text{if } \lambda_1, \lambda_2 \text{ are real numbers;} \\ A \text{ is called parabolic} & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

Moreover, they can be defined as

$$\begin{cases} A \text{ is elliptic} & \text{if } \triangle_A < 0; \\ A \text{ is hyperbolic} & \text{if } \triangle_A > 0; \\ A \text{ is parabolic} & \text{if } \triangle_A = 0. \end{cases} \quad (2.7)$$

where $\triangle_A = (\mathrm{tr}A)^2 - 4\det A$.

Corollary 2.16. *Norm of a 2×2 real matrix, defined as follows :*

$$\begin{aligned} \rho &= \|A\| = \sqrt{|\det A|}, \ \textit{when } A \textit{ is spacelike or timelike matrix,} \\ \rho &= \|A\| = \mathrm{tr}A, \ \textit{when } A \textit{ is lightlike matrix.} \end{aligned}$$

Classification 2×2 matrices can be given with the following table.

A	$\det A > 0$	$\det A = 0$	$\det A < 0$
(trA)^2<4detA	Timelike Elliptic	$\emptyset$	$\emptyset$
(trA)^2=4detA	Timelike Parabolic	Null Parabolic	$\emptyset$
(trA)^2>4detA	Timelike Hyperbolic	Null Hyperbolic	Spacelike Hyperbolic

Using these classifications and De Moivre formulas for hybrid numbers, the roots of a 2×2 matrix can be found easily. Notice that the 2D rotation matrices in the Euclidean, Lorentzian and Galilean plane,

$$\left[\begin{array}{cc} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right], \left[\begin{array}{cc} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{array}\right], \left[\begin{array}{cc} \theta+1 & -\theta \\ \theta & 1-\theta \end{array}\right]$$

are elliptic, hyperbolic and, parabolic matrix, where these matrices correspond to an elliptic, a hyperbolic and, a parabolic hybrid number, respectively.

3. Polar Representations of 2×2 Matrices

In this section, we give the polar representations of 2×2 matrices with respect to their type and character separately. These representations will depend on whether the matrix is elliptic, hyperbolic, and parabolic. On the other hand, according to whether the matrix is spacelike, timelike or lightlike, the polar representation will change. For this reason, we will give the polar representation of a 2×2 matrix with three different subsections.

Definition 3.1. Let $A = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ is a real matrix. Argument of A defined as follows :

i. If A is elliptic such that $\mathrm{tr}A < 0$, then $\arg A = \theta = \pi - \arctan \frac{\sqrt{-\Delta_A}}{|\mathrm{tr}A|}$;

ii. If A is elliptic such that $\mathrm{tr}A > 0$, then $\arg A = \theta = \arctan \frac{\sqrt{-\Delta_A}}{|\mathrm{tr}A|}$;

iii. If A is hyperbolic, then $\arg A = \theta = \ln\left|\dfrac{\mathrm{tr}A + \sqrt{\Delta_A}}{2\rho}\right|$,

iv. If A is parabolic, then $\arg A = \theta = \dfrac{a-d}{|a+d|}$

where $\rho = \sqrt{|\det A|}$, $\Delta_A = (\mathrm{tr}A)^2 - 4\det A$.

After that, throughout the paper, we will use the above formulas for the argument of elliptic, hyperbolic and parabolic matrices.

3.1. Polar Representation of an Elliptic 2×2 Matrix

Theorem 3.2. *If $A = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ is an elliptic matrix, then A can be written in the polar form as*

$$A = \rho \left[\begin{array}{cc} \cos\theta + \frac{a-d}{\sqrt{-\Delta_A}}\sin\theta & \frac{2b}{\sqrt{-\Delta_A}}\sin\theta \\ \frac{2c}{\sqrt{-\Delta_A}}\sin\theta & \cos\theta - \frac{a-d}{\sqrt{-\Delta_A}}\sin\theta \end{array}\right]. \tag{3.1}$$

Proof. If A is elliptic, then we have $\Delta < 0$ and $\det A = ad - bc > 0$. The hybrid number $\varphi^{-1}(A)$ corresponding to the matrix A is

$$\mathbf{W} = \left(\frac{a+d}{2}\right) + \left(\frac{a+b-c-d}{2}\right)\mathbf{i} + \left(\frac{a-d}{2}\right)\varepsilon + \left(\frac{b+c}{2}\right)\mathbf{h}.$$

Therefore, according to Theorem 2.7, we can write $\mathbf{W} = \rho(\cos\theta + \mathbf{U}\sin\theta)$, where

$$\mathbf{U} = \frac{1}{2\sqrt{-\Delta}}\left((a+b-c-d)\,\mathbf{i} + (a-d)\,\varepsilon + (b+c)\,\mathbf{h}\right).$$

since $\rho = \sqrt{\det A}$ and $\mathcal{N}(\mathbf{W}) = \sqrt{-\Delta}$. Thus, using the isomorphism (2.4), we obtain (3.1). □

3.2. Polar Representation of a Hyperbolic 2×2 Matrix

Theorem 3.3. *If* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is a hyperbolic real matrix, then* A *can be written in the polar form as follows :*
i. if A *is timelike, then*

$$A = \xi\rho \begin{bmatrix} \cosh\theta + \frac{(a-d)}{\sqrt{\Delta_A}}\sinh\theta & \frac{2b}{\sqrt{\Delta_A}}\sinh\theta \\ \frac{2c}{\sqrt{\Delta_A}}\sinh\theta & \cosh\theta - \frac{(a-d)}{\sqrt{\Delta_A}}\sinh\theta \end{bmatrix}, \qquad (3.2)$$

ii. if A *is spacelike, then*

$$A = \rho \begin{bmatrix} \sinh\theta + \frac{(a-d)}{\sqrt{\Delta_A}}\cosh\theta & \frac{2b}{\sqrt{\Delta_A}}\cosh\theta \\ \frac{2c}{\sqrt{\Delta_A}}\cosh\theta & \sinh\theta - \frac{a-d}{\sqrt{\Delta_A}}\cosh\theta \end{bmatrix}. \qquad (3.3)$$

where $\xi =$*sign*$(\mathrm{tr}A)$.

Proof. If A is hyperbolic, then we have $\Delta > 0$. Let the hybrid number $\varphi^{-1}(A)$ corresponding to A be $\mathbf{W}$. The matrix A can be spacelike ,timelike or lightlike according to sign of $\det A$. So, from Theorem (2.7), we can write

$$\mathbf{W} = \begin{cases} \pm\rho(\cosh\theta + \mathbf{U}\sinh\theta), & \text{when } \mathbf{W} \text{ is timelike;} \\ \rho(\sinh\theta + \mathbf{U}\cosh\theta), & \text{when } \mathbf{W} \text{ is spacelike;} \end{cases}$$

where

$$\mathbf{V} = \frac{1}{2\sqrt{\Delta}}\left((a+b-c-d)\,\mathbf{i} + (a-d)\,\varepsilon + (b+c)\,\mathbf{h}\right).$$

Scalar parts of these hybrid numbers depend on the

$$S\left(\varphi^{-1}(A)\right) = \frac{a+d}{2} = \frac{\mathrm{tr}A}{2}.$$

On the other hand, scalar part of $\mathbf{W}$ is

$$S(\mathbf{W}) = \pm\rho\cosh\theta.$$

So, if $\mathrm{tr}A > 0$, namely $\xi = \mathrm{sign}(\mathrm{tr}A) = 1$, then we have to write $\rho\cosh\theta$, on the other case, we write $-\rho\cosh\theta$. Thus, using the isomorphism (2.4), we obtain the equalities (3.2) and (3.3). □

Theorem 3.4. *If* $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is a lightlike hyperbolic matrix, then* A *can be written in the polar form as*

$$A=\mathrm{tr}A\begin{bmatrix} \frac{a}{\mathrm{tr}A} & \frac{b}{\mathrm{tr}A} \\ \frac{c}{\mathrm{tr}A} & \frac{d}{\mathrm{tr}A} \end{bmatrix}. \tag{3.4}$$

Proof. If the matrix A is lightlike, hyperbolic, then

$$\det A=0 \text{ and } (\mathrm{tr}A)^2>4\det A=0.$$

So, $\mathrm{tr}A\neq 0$ and the polar form of the hybrid number $\varphi^{-1}(A)$ corresponding to A can be written as

$$\mathbf{W}=\mathrm{tr}A\left(\tfrac{1}{2}+\left(\tfrac{a+b-c-d}{2\mathrm{tr}A}\right)\mathbf{i}+\left(\tfrac{a-d}{2\mathrm{tr}A}\right)\varepsilon+\left(\tfrac{b+c}{2\mathrm{tr}A}\right)\mathbf{h}\right).$$

Therefore, using the isomorphism (2.5) we get the polar form of A as (3.4). □

3.3. Polar Representations of a Parabolic 2×2 Matrix

Theorem 3.5. *If* $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is a timelike parabolic matrix such that* $a\neq d$, *then* A *can be written in the polar form as*

$$A=\tfrac{\mathrm{tr}A}{2}\begin{bmatrix} 1+\xi\theta & \dfrac{\xi 2b\theta}{a-d} \\ \dfrac{2\xi c\theta}{a-d} & 1-\xi\theta \end{bmatrix}$$

where $\theta=\dfrac{a-d}{|a+d|}$ *and* $\xi=sign(\mathrm{tr}A)$.

Proof. The hybrid number corresponding to A is (2.5). If A is parabolic, then we have $(\mathrm{tr}A)^2=4\det A$. Therefore, we get

$$\|\mathbf{W}\|=\sqrt{|\det A|}=\frac{\xi\mathrm{tr}A}{2}$$

where $\xi=sign\,(\mathrm{tr}A)$. Thus, according to the Theorem (2.7), the polar form of the $\mathbf{W}$ is

$$\mathbf{W}=\tfrac{\xi\mathrm{tr}A}{2}\left(\xi+\left(\tfrac{a+b-c-d}{\xi\mathrm{tr}A}\right)\mathbf{i}+\left(\tfrac{a-d}{\xi\mathrm{tr}A}\right)\varepsilon+\left(\tfrac{b+c}{\xi\mathrm{tr}A}\right)\mathbf{h}\right).$$

So, the argument of $\mathbf{W}$ is

$$\theta=\frac{(a-d)/2}{(\xi\mathrm{tr}A)/2}=\frac{a-d}{\xi\mathrm{tr}A}.$$

Therefore, we have $\mathrm{tr}A=\frac{a-d}{\theta\xi}$ and we can write

$$\mathbf{W}=\tfrac{\xi\mathrm{tr}A}{2}\left(\xi+\left(1+\tfrac{(b-c)\theta}{a-d}\right)\mathbf{i}+\theta\varepsilon+\left(\tfrac{(b+c)\theta}{a-d}\right)\mathbf{h}\right)$$

Thus, using (2.4) and $\mathrm{tr}A = a+d = \frac{a-d}{\theta\xi}$, we obtain,

$$A = \frac{\mathrm{tr}A}{2}\begin{bmatrix} \xi\theta+1 & \dfrac{2b\xi\theta}{a-d} \\ \dfrac{2c\xi\theta}{a-d} & 1-\xi\theta \end{bmatrix}.$$

□

Theorem 3.6. *If* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is a lightlike parabolic matrix, then* $\det A = 0$ *and* $\mathrm{tr}A = 0$*. So, A can be written as*

$$\begin{bmatrix} a & -\dfrac{a^2}{c} \\ c & -a \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix},$$

depending on whether $c \neq 0$ or $c = 0$, respectively. A parabolic lightlike matrix is a nilpotent matrix. That is, $A^n = 0$, for all $n \in \mathbb{N}$.

Proof. If A is a parabolic null matrix, then $\det A = 0$ and $trA = 0$. It means that $a+d = 0$ and $ad-bc = 0$. So, $d = -a$, $bc = -a^2$. If $c = 0$, then $a = d = 0$. In the case $c \neq 0$, we obtain $b = -a^2/c$. □

Theorem 3.7. *If the matrix* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is a timelike parabolic matrix with $a = d$. So, A can be written as*

$$A = a\begin{bmatrix} 1 & 0 \\ c/a & 1 \end{bmatrix}, \ A = a\begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix} \text{ or } A = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.5}$$

according to whether $b = 0$, $c = 0$ or $b = c = 0$, respectively.

Proof. If A is parabolic, then $(\mathrm{tr}A)^2 = 4\det A$. So, in the case $a = d$, we find that $a^2 = a^2 - bc$ and $bc = 0$. □

4. De Moivre's Formula for 2×2 Matrices

In this section, using the polar forms, we can express De Moivre's formulas for the 2×2 matrices. Here, the De Moivre's formulas also change with respect to the types of the matrices.

4.1. De Moivre's formula for Elliptic Matrices

Theorem 4.1. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be an elliptic real matrix whose polar representation is*

$$A = \rho\begin{bmatrix} \cos\theta + \frac{a-d}{\sqrt{-\Delta_A}}\sin\theta & \frac{2b}{\sqrt{-\Delta_A}}\sin\theta \\ \frac{2c}{\sqrt{-\Delta_A}}\sin\theta & \cos\theta - \frac{a-d}{\sqrt{-\Delta_A}}\sin\theta \end{bmatrix},$$

then A^n has the form

$$A^n=\rho^n\begin{bmatrix}\cos(n\theta)+\frac{a-d}{\sqrt{-\Delta_A}}\sin(n\theta) & \frac{2b}{\sqrt{-\Delta_A}}\sin(n\theta)\\ \frac{2c}{\sqrt{-\Delta_A}}\sin(n\theta) & \cos(n\theta)-\frac{a-d}{\sqrt{-\Delta_A}}\sin(n\theta)\end{bmatrix} \tag{4.1}$$

for $n\in\mathbb{Z}$.

Proof. It can be proved by induction. It is true for $n=1$. Assume that (4.1) holds for $n=k$. Then, using $\Delta_A=a^2-2ad+d^2+4bc$, we obtain A^{k+1} as

$$\rho^{k+1}\begin{bmatrix}\cos(k+1)\theta+\frac{a-d}{\sqrt{-\Delta_A}}\sin(k+1)\theta & \frac{2b}{\sqrt{-\Delta_A}}\sin(k+1)\theta\\ \frac{2c}{\sqrt{-\Delta_A}}\sin(k+1)\theta & \cos(k+1)\theta-\frac{a-d}{\sqrt{-\Delta_A}}\sin(k+1)\theta\end{bmatrix}.$$

Moreover, since

$$A^{-1}=\rho^{-1}\begin{bmatrix}\cos\theta-\frac{a-d}{\sqrt{-\Delta_A}}\sin\theta & \frac{-2b}{\sqrt{-\Delta}}\sin\theta\\ \frac{-2c}{\sqrt{-\Delta}}\sin\theta & \cos\theta+\frac{a-d}{\sqrt{-\Delta_A}}\sin\theta\end{bmatrix},$$

we have

$$A^{-n}=\rho^{-n}\begin{bmatrix}\cos(n\theta)-\frac{a-d}{\sqrt{-\Delta_A}}\sin(n\theta) & \frac{-2b}{\sqrt{-\Delta}}\sin(n\theta)\\ \frac{-2c}{\sqrt{-\Delta}}\sin(n\theta) & \cos(n\theta)+\frac{a-d}{\sqrt{-\Delta_A}}\sin(n\theta)\end{bmatrix}.$$

So, the formula holds for all integers. □

Corollary 4.2. *If $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ is an elliptic real matrix, then A^n is also an elliptic matrix.*

Proof. A^n is an elliptic matrix if and only if $(\mathrm{tr}A^n)^2<4\det A^n$. So, using (4.1), we obtain $\mathrm{tr}A^n=2\rho^n\cos^2 n\theta$ and $\det A^n=\rho^{2n}$. Therefore, we have $\mathrm{tr}A^n=4\left(\cos^2 n\theta\right)\det A^n$ and $(\mathrm{tr}A^n)^2<4\det A^n$, since $0<\cos^2\theta\le 1$. □

4.2. De Moivre's formula for Hyperbolic Matrices

Theorem 4.3. *If $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ is a timelike hyperbolic real matrix whose polar representation is* (3.2)*, then*

$$A^n=\xi^n\rho^n\begin{bmatrix}\cosh(n\theta)+\frac{(a-d)}{\sqrt{\Delta_A}}\sinh(n\theta) & \frac{2b}{\sqrt{\Delta_A}}\sinh(n\theta)\\ \frac{2c}{\sqrt{\Delta_A}}\sinh(n\theta) & \cosh(n\theta)-\frac{(a-d)}{\sqrt{\Delta_A}}\sinh(n\theta)\end{bmatrix} \tag{4.2}$$

for $n\in\mathbb{Z}$, where $\xi=$sign$(\mathrm{tr}A)$.

Proof. It can be proved by induction similar to proof of Theorem 4.1 and using the equality $\Delta_A=a^2-2ad+d^2+4bc$. □

Theorem 4.4. *Let $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ be a spacelike hyperbolic real matrix whose polar representation is* (3.3)*.*

i. If n is an even integer, then A^n is a timelike matrix and

$$A^n = \xi^n \rho^n \begin{bmatrix} \cosh(n\theta) + \frac{a-d}{\sqrt{\Delta_A}} \sinh(n\theta) & \frac{2b}{\sqrt{\Delta_A}} \sinh(n\theta) \\ \frac{2c}{\sqrt{\Delta_A}} \sinh(n\theta) & \cosh(n\theta) - \frac{a-d}{\sqrt{\Delta_A}} \sinh(n\theta) \end{bmatrix} \tag{4.3}$$

ii. If n is an odd integer, then A^n is a spacelike matrix and

$$A^n = \xi^n \rho^n \begin{bmatrix} \sinh(n\theta) + \frac{a-d}{\sqrt{\Delta_A}} \cosh(n\theta) & \frac{2b}{\sqrt{\Delta_A}} \cosh(n\theta) \\ \frac{2c}{\sqrt{\Delta_A}} \cosh(n\theta) & \sinh(n\theta) - \frac{a-d}{\sqrt{\Delta_A}} \cosh(n\theta) \end{bmatrix} \tag{4.4}$$

where $\xi = sign(\mathrm{tr}A)$, $\rho = \sqrt{|\det A|}$, $\theta = \ln\left|\frac{\mathrm{tr}A + \sqrt{\Delta_A}}{2\sqrt{|\det A|}}\right|$ *and* $\Delta_A = (\mathrm{tr}A)^2 - 4\det A$.

Proof. Hybrid product of two spacelike hybrid numbers is a timelike hybrid number. So, if A is a spacelike matrix and n is an even number, A^n will be a timelike matrix. Also, if n is an odd number, then A^n will be a spacelike matrix. Let polar form of the spacelike hyperbolic matrix A be

$$\xi\rho \begin{bmatrix} \sinh\theta + \frac{a-d}{\sqrt{\Delta_A}} \cosh\theta & \frac{2b}{\sqrt{\Delta_A}} \cosh\theta \\ \frac{2c}{\sqrt{\Delta_A}} \cosh\theta & \sinh\theta - \frac{a-d}{\sqrt{\Delta_A}} \cosh\theta \end{bmatrix}.$$

Then, using the equality $\Delta_A = a^2 - 2ad + d^2 + 4bc$, we obtain

$$\begin{aligned} A^2 &= \rho^2 \begin{bmatrix} \cosh(2\theta) + \frac{a-d}{\sqrt{\Delta_A}} \sinh(2\theta) & \frac{2b}{\sqrt{\Delta_A}} \sinh(2\theta) \\ \frac{2c}{\sqrt{\Delta_A}} \sinh(2\theta) & \cosh(2\theta) - \frac{a-d}{\sqrt{\Delta_A}} \sinh(2\theta) \end{bmatrix}, \\ A^3 &= \xi\rho^3 \begin{bmatrix} \sinh 3\theta + \frac{a-d}{\sqrt{\Delta_A}} \cosh 3\theta & \frac{2b}{\sqrt{\Delta_A}} \cosh 3\theta \\ \frac{2c}{\sqrt{\Delta_A}} \cosh 3\theta & \sinh 3\theta - \frac{a-d}{\sqrt{\Delta_A}} \cosh 3\theta \end{bmatrix}. \end{aligned}$$

Therefore, it can be proved by induction. Let k is an even number and (4.3) is true for $n = k$. Then, $k + 1$ is odd. So, we get A^{k+1} as,

$$\xi\rho^{k+1} \begin{bmatrix} \sinh(k+1)\theta + \frac{a-d}{\sqrt{\Delta_A}} \cosh(k+1)\theta & \frac{2b}{\sqrt{\Delta_A}} \cosh(k+1)\theta \\ \frac{2c}{\sqrt{\Delta_A}} \cosh(k+1)\theta & \sinh(k+1)\theta - \frac{a-d}{\sqrt{\Delta_A}} \cosh(k+1)\theta \end{bmatrix}.$$

Similarly, if k is an odd number, $k + 1$ is even and A^{k+1} is

$$\rho^{k+1} \begin{bmatrix} \cosh(k+1)\theta + \frac{(a-d)}{\sqrt{\Delta_A}} \sinh(k+1)\theta & \frac{2b}{\sqrt{\Delta_A}} \sinh(k+1)\theta \\ \frac{2c}{\sqrt{\Delta_A}} \sinh(k+1)\theta & \cosh(k+1)\theta - \frac{(a-d)}{\sqrt{\Delta_A}} \sinh(k+1)\theta \end{bmatrix}.$$

□

Theorem 4.5. *If* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is a spacelike hyperbolic real matrix such that* $\mathrm{tr}A = 0$, *then* A^n *is*

$$A^n = \begin{cases} \rho^{n-1}A & \text{when } n \text{ is odd;} \\ \rho^n I & \text{when } n \text{ is even.} \end{cases} \tag{4.5}$$

where A^n is a parabolic matrix.

Proof. Let A be a spacelike hyperbolic matrix such that $\mathrm{tr}A = 0$, then we have $\rho = \sqrt{-\det A}$ and $\Delta_A = -4\det A > 0$. So, we get

$$\theta = \ln\left|\frac{\mathrm{tr}A+\sqrt{\Delta_A}}{2\sqrt{|\det A|}}\right| = \ln\left|\frac{2\sqrt{-\det A}}{2\sqrt{-\det A}}\right| = \ln 1 = 0.$$

Thus, according to (4.3) and (4.4), we obtain

$$A^n = \rho^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } A^n = \rho^{n-1} \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

for n is even or odd, respectively. □

Corollary 4.6. *Let A be a 2×2 spacelike hyperbolic matrix such that $\mathrm{tr}A = 0$. A^n is a parabolic matrix if and only if n is an even number.*

Proof. A^n is a parabolic matrix if and only if $\Delta_{A^n} = 0$. According to (4.5), $\Delta_{A^n} = -4\rho^{2(n-1)}\det A$ for n is odd number and $\Delta_{A^n} = 0$ for n is even number. We know that $\det A < 0$ for a spacelike matrix, then $\det A \neq 0$ and $\Delta_{A^n} \neq 0$ for n is odd. So, A^n is a parabolic matrix if and only if n is an even number. □

Corollary 4.7. *Let A be a 2×2 lightlike parabolic matrix such that $\mathrm{tr}A = 0$. A^n is a parabolic matrix if and only if n is an even number.*

Theorem 4.8. *Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a lightlike hyperbolic real matrix whose polar representation is*

$$A = \mathrm{tr}A \begin{bmatrix} \frac{a}{\mathrm{tr}A} & \frac{b}{\mathrm{tr}A} \\ \frac{c}{\mathrm{tr}A} & \frac{d}{\mathrm{tr}A} \end{bmatrix}.$$

Then, we have

$$A^n = (\mathrm{tr}A)^n \begin{bmatrix} \frac{a}{\mathrm{tr}A} & \frac{b}{\mathrm{tr}A} \\ \frac{c}{\mathrm{tr}A} & \frac{d}{\mathrm{tr}A} \end{bmatrix}. \tag{4.6}$$

for $n \in \mathbb{Z}^+$.

Proof. We know that $\mathrm{tr}A \neq 0$ for a lightlike hyperbolic matrix A. Let's prove it by induction. Assume that (4.6) is true for $n = k$. Then,

$$A^{k+1} = (\mathrm{tr}A)^{n+1} \begin{bmatrix} \frac{a}{\mathrm{tr}A} & \frac{b}{\mathrm{tr}A} \\ \frac{c}{\mathrm{tr}A} & \frac{d}{\mathrm{tr}A} \end{bmatrix}^2 = \frac{(\mathrm{tr}A)^{n+1}}{(\mathrm{tr}A)^2} \begin{bmatrix} a^2+bc & b\,(a+d) \\ c\,(a+d) & d^2+bc \end{bmatrix}.$$

On the other hand, we know that $\det A = 0$, so, $ad = bc$ for the lightlike matrix A. If we use this equality, we get

$$A^{k+1} = \frac{(\mathrm{tr}A)^{n+1}}{(\mathrm{tr}A)^2} \begin{bmatrix} a\mathrm{tr}A & b\mathrm{tr}A \\ c\mathrm{tr}A & d\mathrm{tr}A \end{bmatrix} = (\mathrm{tr}A)^{n+1} \begin{bmatrix} \frac{a}{\mathrm{tr}A} & \frac{b}{\mathrm{tr}A} \\ \frac{c}{\mathrm{tr}A} & \frac{d}{\mathrm{tr}A} \end{bmatrix}.$$

□

Corollary 4.9. *If A is a lightlike hyperbolic matrix, then A^n is also a lightlike hyperbolic matrix.*

Proof. If A is a lightlike hyperbolic matrix, then $\det A = 0$ and $\mathrm{tr} A \neq 0$. So, using (4.6), we obtain $\det A^n = 0$. That is, A^n is also a lightlike hyperbolic matrix. $\square$

Corollary 4.10. *If* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is a hyperbolic real matrix such that* $\mathrm{tr} A \neq 0$*, then* A^n *is also an hyperbolic matrix.*

Proof. A^n is hyperbolic if and only if $(\mathrm{tr}\,(A^n))^2 > 4 \det A^n$. If A is hyperbolic, A^n can be one of the equalities (4.2), (4.3), (4.4) and, (4.6). If A^n is (4.2) or (4.3), we obtain

$$\mathrm{tr}\,(A^n) = 2\xi^n \rho^n \cosh n\theta \text{ and } \det A^n = \rho^{2n}.$$

Therefore, we have $(\mathrm{tr}\,(A^n))^2 = 4 \cosh^2 n\theta \det A^n$. If A^n is (4.4), we have

$$\mathrm{tr}\,(A^n) = 2\xi^n \rho^n \sinh n\theta \text{ and } \det A^n = -\rho^{2n}.$$

and $(\mathrm{tr}\,(A^n))^2 = -4 \sinh^2 n\theta \,(\det A^n)$. Finally, if A^n is in the form (4.6), then

$$\mathrm{tr}\,(A^n) = (\mathrm{tr} A)^n \text{ and } \det A^n = 0.$$

Thus, we see that the inequality $(\mathrm{tr} A)^2 > 4 \det A^n$ is true for all cases of A^n, if $\mathrm{tr} A \neq 0$. $\square$

4.3. De Moivre's formula for Parabolic Matrices

Theorem 4.11. *If* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a \neq d$ *is a timelike parabolic matrix whose polar representation is*

$$A = \tfrac{\mathrm{tr} A}{2} \begin{bmatrix} \xi\theta + 1 & \dfrac{2b\xi\theta}{a-d} \\ \dfrac{2c\xi\theta}{a-d} & 1 - \xi\theta \end{bmatrix},$$

then

$$A^n = \left(\tfrac{\mathrm{tr} A}{2}\right)^n \begin{bmatrix} \xi n\theta + 1 & \dfrac{2bn\xi\theta}{a-d} \\ \dfrac{2cn\xi\theta}{a-d} & 1 - \xi n\theta \end{bmatrix} \tag{4.7}$$

for $n \in \mathbb{Z}$ *where* $\theta = \frac{a-d}{|a+d|}$ *and* $\xi = $*sign*$(\mathrm{tr} A)$.

Proof. Assume that (4.7) is true for $n = k$. Then, we get

$$A^{k+1} = \left(\tfrac{\mathrm{tr} A}{2}\right)^{k+1} \begin{bmatrix} \xi\theta + 1 & \dfrac{2b\xi\theta}{a-d} \\ \dfrac{2c\xi\theta}{a-d} & 1 - \xi\theta \end{bmatrix} \begin{bmatrix} \xi k\theta + 1 & \dfrac{2b\xi k\theta}{a-d} \\ \dfrac{2c\xi k\theta}{a-d} & 1 - \xi k\theta \end{bmatrix}.$$

On the other hand, in a parabolic timelike matrix, we know that the equality $(\mathrm{tr} A)^2 = 4 \det A \neq 0$ satisfies. So, we have $(a-d)^2 = -4bc$ and we obtain

$$A^{k+1} = \left(\tfrac{\mathrm{tr} A}{2}\right)^{k+1} \begin{bmatrix} 1 + (k+1)\,\theta\xi & \dfrac{2b\xi\,(k+1)\,\theta}{a-d} \\ \dfrac{2c\xi\,(k+1)\,\theta}{a-d} & 1 - (k+1)\,\theta\xi \end{bmatrix}.$$

□

Theorem 4.12. *If* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is a timelike parabolic matrix with* $a = d$*, then* A^n *is*

$$a^n \begin{bmatrix} 1 & 0 \\ nc/a & 1 \end{bmatrix}, \ a^n \begin{bmatrix} 1 & nb/a \\ 0 & 1 \end{bmatrix}$$
$$\text{or } a^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

with respect to whether $b = 0$*,* $c = 0$ *or* $b = c = 0$*, respectively.*

Proof. If A is parabolic, then $(\mathrm{tr}A)^2 = 4\det A$. So, in the case $a = d$, we find that $a^2 = a^2 - bc$ and $bc = 0$. Thus, it is clear from the Theorem 3.7. □

Corollary 4.13. *If* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is a timelike parabolic matrix, then* A^n *is also a timelike parabolic matrix.*

Proof. We know that, A is a timelike parabolic matrix if and only if

$$(\mathrm{tr}A)^2 = 4\det A \neq 0.$$

Therefore, we obtain

$$\det A^n = \left(\tfrac{\mathrm{tr}A}{2}\right)^{2n} \text{ and } \mathrm{tr}A^n = 2\left(\tfrac{\mathrm{tr}A}{2}\right)^n.$$

using (4.7), Thus, we see that the equality $(\mathrm{tr}A^n) = 4\det A^n$ satisfies. It means that A^n is also a timelike parabolic matrix. □

Theorem 4.14. *If* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *is a lightlike parabolic real matrix, then*

$$A^n = 0$$

for all $n \in \mathbb{Z}^+$.

Proof. See Theorem (3.6). □

5. Roots of a 2×2 Matrix

In this section we study n-th roots of a 2 by 2 real matrix, considering the De Moivre's formulas given above.

5.1. n-th Roots of an Elliptic Matrix

Theorem 5.1. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be an elliptic real matrix whose polar representation is*

$$A = \rho \begin{bmatrix} \cos\theta + \frac{(a-d)}{\sqrt{-\Delta_A}} \sin\theta & \frac{2b}{\sqrt{-\Delta_A}} \sin\theta \\ \frac{2c}{\sqrt{-\Delta_A}} \sin\theta & \cos\theta - \frac{(a-d)}{\sqrt{-\Delta_A}} \sin\theta \end{bmatrix},$$

then the n*-th roots of the matrix* A *are*

$$\rho^{1/n} \begin{bmatrix} \cos\frac{\theta+2\pi k}{n} + \frac{(a-d)}{\sqrt{-\Delta_A}} \sin\frac{\theta+2\pi k}{n} & \frac{2b}{\sqrt{-\Delta_A}} \sin\frac{\theta+2\pi k}{n} \\ \frac{2c}{\sqrt{-\Delta_A}} \sin\frac{\theta+2\pi k}{n} & \cos\frac{\theta+2\pi k}{n} - \frac{(a-d)}{\sqrt{-\Delta_A}} \sin\frac{\theta+2\pi k}{n} \end{bmatrix} \tag{5.1}$$

where $k = 0, 1, 2, ..., n-1.$

Proof. Let the matrix X be an n-th root of A. Then we have $X^n = A$. Considering the Theorem (4.1) and the fact that positive integer power of an elliptic matrix is an elliptic matrix, X is in the form

$$X = \rho_x \begin{bmatrix} \cos\beta + \frac{(a-d)}{\sqrt{-\Delta_A}} \sin\beta & \frac{2b}{\sqrt{-\Delta_A}} \sin\beta \\ \frac{2c}{\sqrt{-\Delta_A}} \sin\beta & \cos\beta - \frac{(a-d)}{\sqrt{-\Delta_A}} \sin\beta \end{bmatrix}.$$

So, we have

$$X^n = \rho_x^n \begin{bmatrix} \cos n\beta + \frac{(a-d)}{\sqrt{-\Delta_A}} \sin n\beta & \frac{2b}{\sqrt{-\Delta_A}} \sin n\beta \\ \frac{2c}{\sqrt{-\Delta_A}} \sin n\beta & \cos n\beta - \frac{(a-d)}{\sqrt{-\Delta_A}} \sin n\beta \end{bmatrix}.$$

Therefore, from the equality $X^n = A$ and equality of matrices, we obtain, $\beta = \frac{\theta+2k\pi}{n}$ and $\rho_x = \sqrt[n]{\rho}$ for $k = 0, 1, 2, ..., n-1.$ □

Example 1. Let's find n-th roots of the elliptic matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$. The polar form of A is

$$A = \sqrt{3} \begin{bmatrix} \cos\theta & \sqrt{2}\sin\theta \\ -\frac{\sqrt{2}}{2}\sin\theta & \cos\theta \end{bmatrix}$$

where $\theta = \arctan\sqrt{2}$. According to Theorem (5.1), we obtain

$$\sqrt[n]{A} = 3^{1/2n} \begin{bmatrix} \cos\dfrac{\theta+2\pi k}{n} & \sqrt{2}\sin\dfrac{\theta+2\pi k}{n} \\ -\dfrac{\sqrt{2}}{2}\sin\dfrac{\theta+2\pi k}{n} & \cos\dfrac{\theta+2\pi k}{n} \end{bmatrix}$$

where $k = 0, 1, 2, ..., n-1$. For instances, third roots of A are

$$\sqrt[3]{A} = 3^{1/6} \begin{bmatrix} \cos\frac{\theta+2\pi k}{3} & \sqrt{2}\sin\frac{\theta+2\pi k}{3} \\ -\frac{\sqrt{2}}{2}\sin\frac{\theta+2\pi k}{3} & \cos\frac{\theta+2\pi k}{3} \end{bmatrix}, \text{ for } k = 0, 1, 2.$$

So, the roots $\sqrt[3]{A}$ are

$$\begin{aligned} A_1 &= 3^{1/6}\begin{bmatrix} \cos\frac{\left(\arctan\sqrt{2}\right)}{3} & \sqrt{2}\sin\frac{\left(\arctan\sqrt{2}\right)}{3} \\ -\frac{\sqrt{2}}{2}\sin\frac{\left(\arctan\sqrt{2}\right)}{3} & \cos\frac{\left(\arctan\sqrt{2}\right)}{3} \end{bmatrix} \\ &\simeq \begin{bmatrix} 1.1406 & 0.53174 \\ -0.26587 & 1.1406 \end{bmatrix}. \end{aligned}$$

$$\begin{aligned} A_2 &= 3^{1/6}\begin{bmatrix} \cos\frac{2\pi+\arctan\sqrt{2}}{3} & \sqrt{2}\sin\frac{2\pi+\arctan\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{2}\sin\frac{2\pi+\arctan\sqrt{2}}{3} & \cos\frac{2\pi+\arctan\sqrt{2}}{3} \end{bmatrix} \\ &\simeq \begin{bmatrix} -0.8959 & 1.131 \\ -0.56551 & -0.8959 \end{bmatrix}. \end{aligned}$$

and

$$\begin{aligned} A_3 &= 3^{1/6}\begin{bmatrix} \cos\frac{4\pi+\arctan\sqrt{2}}{3} & \sqrt{2}\sin\frac{4\pi+\arctan\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{2}\sin\frac{4\pi+\arctan\sqrt{2}}{3} & \cos\frac{4\pi+\arctan\sqrt{2}}{3} \end{bmatrix} \\ &\simeq \begin{bmatrix} -0.24466 & -1.6628 \\ 0.83138 & -0.24466 \end{bmatrix}. \end{aligned}$$

Corollary 5.2. *If A is an elliptic matrix, then there are n matrices X satisfying the equality $X^n = A$. So, an elliptic matrix has 2 square roots.*

5.2. n-th Roots of a Hyperbolic Matrix

Theorem 5.3. *Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a timelike hyperbolic real matrix whose polar representation is*

$$A = \xi_A\rho\begin{bmatrix} \cosh\theta + \frac{(a-d)}{\sqrt{\Delta_A}}\sinh\theta & \frac{2b}{\sqrt{\Delta_A}}\sinh\theta \\ \frac{2c}{\sqrt{\Delta_A}}\sinh\theta & \cosh\theta - \frac{(a-d)}{\sqrt{\Delta_A}}\sinh\theta \end{bmatrix}.$$

i. If n is even number and $\xi_A = 1$, then n-th roots of A are

$$\sqrt[n]{A} = \pm\rho^{1/n}\begin{bmatrix} \cosh\frac{\theta}{n} + \frac{(a-d)}{\sqrt{\Delta_A}}\sinh\frac{\theta}{n} & \frac{2b}{\sqrt{\Delta_A}}\sinh\frac{\theta}{n} \\ \frac{2c}{\sqrt{\Delta_A}}\sinh\frac{\theta}{n} & \cosh\frac{\theta}{n} - \frac{(a-d)}{\sqrt{\Delta_A}}\sinh\frac{\theta}{n} \end{bmatrix} \tag{5.2}$$

and

$$\sqrt[n]{A} = \pm\rho^{1/n}\begin{bmatrix} \sinh\frac{\theta}{n} + \frac{(a-d)}{\sqrt{\Delta_A}}\cosh\frac{\theta}{n} & \frac{2b}{\sqrt{\Delta_A}}\cosh\frac{\theta}{n} \\ \frac{2c}{\sqrt{\Delta_A}}\cosh\frac{\theta}{n} & \sinh\frac{\theta}{n} - \frac{(a-d)}{\sqrt{\Delta_A}}\cosh\frac{\theta}{n} \end{bmatrix}. \tag{5.3}$$

ii. If n is odd number, then n-th roots of A are

$$\sqrt[n]{A} = \xi_A\rho^{1/n}\begin{bmatrix} \cosh\frac{\theta}{n} + \frac{(a-d)}{\sqrt{\Delta_A}}\sinh\frac{\theta}{n} & \frac{2b}{\sqrt{\Delta_A}}\sinh\frac{\theta}{n} \\ \frac{2c}{\sqrt{\Delta_A}}\sinh\frac{\theta}{n} & \cosh\frac{\theta}{n} - \frac{(a-d)}{\sqrt{\Delta_A}}\sinh\frac{\theta}{n} \end{bmatrix}. \tag{5.4}$$

Proof. Let the matrix X be an n-th root of A. Then we have $X^n = A$. X can be one of the forms (3.2), (3.3) or (3.4). If X in the form (3.2), that is X is a timelike hyperbolic matrix, we can write as

$$X = \xi_x \rho_x \begin{bmatrix} \cosh\beta + \frac{(a-d)}{\sqrt{\Delta_A}}\sinh\beta & \frac{2b}{\sqrt{\Delta_A}}\sinh\beta \\ \frac{2c}{\sqrt{\Delta_A}}\sinh\beta & \cosh\beta - \frac{(a-d)}{\sqrt{\Delta_A}}\sinh\beta \end{bmatrix}.$$

So, considering the Theorem (4.3), we have

$$X^n = \xi_x^n \rho_x^n \begin{bmatrix} \cosh(n\beta) + \frac{(a-d)}{\sqrt{\Delta_A}}\sinh(n\beta) & \frac{2b}{\sqrt{\Delta_A}}\sinh(n\beta) \\ \frac{2c}{\sqrt{\Delta_A}}\sinh(n\beta) & \cosh(n\beta) - \frac{(a-d)}{\sqrt{\Delta_A}}\sinh(n\beta) \end{bmatrix}.$$

Therefore, from the equality $X^n = A$ and equality of matrices, we obtain,

$$\beta = \tfrac{\theta}{n} \text{ and } \xi_x^n \rho_x^n = \xi\rho.$$

If n is even, then we have $\rho_x^n = \xi\rho$ and it has a solution if and only if $\xi = 1$. If n is odd number, then $\xi_X = \xi$ and $\rho_x = \sqrt[n]{\rho}$.

If X in the form (3.3), then X is in the form

$$X = \xi_X \rho_X \begin{bmatrix} \sinh\beta + \frac{(a-d)}{\sqrt{\Delta_A}}\cosh\beta & \frac{2b}{\sqrt{\Delta_A}}\cosh\beta \\ \frac{2c}{\sqrt{\Delta_A}}\cosh\beta & \sinh\beta - \frac{(a-d)}{\sqrt{\Delta_A}}\cosh\beta \end{bmatrix}.$$

So, if n is odd, we have no solution for $X^n = A$, since

$$X^n = \xi_x^n \rho_x^n \begin{bmatrix} \sinh(n\beta) + \frac{(a-d)}{\sqrt{\Delta_A}}\cosh(n\beta) & \frac{b}{\sqrt{\Delta_A}}\cosh(n\beta) \\ \frac{2c}{\sqrt{\Delta_A}}\cosh(n\beta) & \sinh(n\beta) - \frac{(a-d)}{\sqrt{\Delta_A}}\cosh(n\beta) \end{bmatrix}.$$

If n is even number, then

$$X^n = \rho_x^n \begin{bmatrix} \cosh(n\beta) + \frac{(a-d)}{\sqrt{\Delta_A}}\sinh(n\beta) & \frac{2b}{\sqrt{\Delta_A}}\sinh(n\beta) \\ \frac{2c}{\sqrt{\Delta_A}}\sinh(n\beta) & \cosh(n\beta) - \frac{(a-d)}{\sqrt{\Delta_A}}\sinh(n\beta) \end{bmatrix}.$$

Thus, we have another solution where $\beta = \frac{\theta}{n}$ and $\rho_x^n = \xi\rho$ for $\xi = 1$. At last, we can see that there is no solution, if X is in the form (3.4). □

Now, let's give two examples for n is odd or even.

Example 2. Let's find $\sqrt[3]{A}$ for the matrix

$$A = \begin{bmatrix} -13 & -21 \\ 14 & 22 \end{bmatrix}.$$

A is a timelike hyperbolic matrix, since $\Delta_A = 9^2 - 32 = 49$ and $\det A = 8 > 0$. Then, the polar form of A is

$$A = \sqrt{8} \begin{bmatrix} \cosh\theta - 5\sinh\theta & -6\sinh\theta \\ 4\sinh\theta & \cosh\theta + 5\sinh\theta \end{bmatrix}$$

where $\theta=\ln 2\sqrt{2}$. Using the Theorem (5.3) for odd number n, we find $\sqrt[3]{A}$ as

$$\begin{aligned}\sqrt[3]{A} &= 8^{1/6}\begin{bmatrix}\cosh\frac{\ln 2\sqrt{2}}{3}-5\sinh\frac{\ln 2\sqrt{2}}{3} & -6\sinh\frac{\ln 2\sqrt{2}}{3}\\ 4\sinh\frac{\ln 2\sqrt{2}}{3} & \cosh\frac{\ln 2\sqrt{2}}{3}+5\sinh\frac{\ln 2\sqrt{2}}{3}\end{bmatrix}\\ &= \begin{bmatrix}-1 & -3\\ 2 & 4\end{bmatrix}.\end{aligned}$$

Example 3. Let's find $\sqrt[4]{A}$ for the matrix

$$A=\begin{bmatrix}11 & 10\\ 5 & 6\end{bmatrix}.$$

A is a timelike hyperbolic matrix, since $\Delta_A=(17)^2-64=225$ and $\det A=16>0$. Then, the polar form of A is

$$A=4\begin{bmatrix}\cosh\theta+\dfrac{1}{3}\sinh\theta & \dfrac{4}{3}\sinh\theta\\ \dfrac{2}{3}\sinh\theta & \cosh\theta-\dfrac{1}{3}\sinh\theta\end{bmatrix}$$

where $\theta=\ln 4$. Therefore, since $n=4$ and $\xi_A=1$, there are four roots $\sqrt[4]{A}$, and these are,

$$\begin{aligned}\sqrt[n]{A} &= \pm\sqrt[4]{4}\begin{bmatrix}\cosh\frac{\ln 4}{4}+\dfrac{1}{3}\sinh\frac{\ln 4}{4} & \dfrac{4}{3}\sinh\frac{\ln 4}{4}\\ \dfrac{2}{3}\sinh\frac{\ln 4}{4} & \cosh\frac{\ln 4}{4}-\dfrac{1}{3}\sinh\frac{\ln 4}{4}\end{bmatrix}\\ A_{1,2} &\simeq \pm\frac{1}{3}\begin{bmatrix}5 & 2\\ 1 & 4\end{bmatrix}.\end{aligned}$$

$$\begin{aligned}\sqrt[n]{A} &= \pm\sqrt[4]{4}\begin{bmatrix}\sinh\frac{\ln 4}{4}+\dfrac{1}{3}\cosh\frac{\ln 4}{4} & \dfrac{4}{3}\cosh\frac{\ln 4}{4}\\ \dfrac{2}{3}\cosh\frac{\ln 4}{4} & \sinh\frac{\ln 4}{4}-\dfrac{1}{3}\cosh\frac{\ln 4}{4}\end{bmatrix}\\ A_{3,4} &= \pm\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix}.\end{aligned}$$

Theorem 5.4. *Let* $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ *be a spacelike hyperbolic matrix whose polar representation is,*

$$A=\rho\begin{bmatrix}\sinh\theta+\frac{(a-d)}{\sqrt{\Delta_A}}\cosh\theta & \frac{2b}{\sqrt{\Delta_A}}\cosh\theta\\ \frac{2c}{\sqrt{\Delta_A}}\cosh\theta & \sinh\theta-\frac{a-d}{\sqrt{\Delta_A}}\cosh\theta\end{bmatrix}.$$

i. If n is odd number, then $\sqrt[n]{A}$ is

$$\sqrt[n]{A}=\rho^{1/n}\begin{bmatrix}\sinh\frac{\theta}{n}+\frac{(a-d)}{\sqrt{\Delta_A}}\cosh\frac{\theta}{n} & \frac{2b}{\sqrt{\Delta_A}}\cosh\frac{\theta}{n}\\ \frac{2c}{\sqrt{\Delta_A}}\cosh\frac{\theta}{n} & \sinh\frac{\theta}{n}-\frac{(a-d)}{\sqrt{\Delta_A}}\cosh\frac{\theta}{n}\end{bmatrix}. \tag{5.5}$$

ii. If n is even number, then there is no n-th root of A.

Proof. It can be proved similar to Theorem (5.3). □

Example 4. Let's find $\sqrt[5]{A}$ for the matrix $A=\begin{bmatrix} -13 & 5 \\ -5 & 2 \end{bmatrix}$. A is a spacelike hyperbolic matrix, since $\Delta_A=(-11)^2+4=125$ and $\det A=-1<0$.

$$A=\begin{bmatrix} \sinh\theta-\frac{3\sqrt{5}}{5}\cosh\theta & \frac{2\sqrt{5}}{5}\cosh\theta \\ -\frac{2\sqrt{5}}{5}\cosh\theta & \sinh\theta+\frac{3\sqrt{5}}{5}\cosh\theta \end{bmatrix}$$

where $\theta=\ln\left(\frac{5\sqrt{5}-11}{2}\right)$. So, the only 5th root of A is

$$\begin{aligned} \sqrt[5]{A} &= \begin{bmatrix} \sinh\frac{\theta}{5}-\frac{3\sqrt{5}}{5}\cosh\frac{\theta}{5} & \frac{2\sqrt{5}}{5}\cosh\frac{\theta}{5} \\ -\frac{2\sqrt{5}}{5}\cosh\frac{\theta}{5} & \sinh\dfrac{\theta}{5}+\frac{3\sqrt{5}}{5}\cosh\frac{\theta}{5} \end{bmatrix} \\ &= \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix}. \end{aligned}$$

Example 5. There is no any root $\sqrt[4]{A}$ for the spacelike hyperbolic matrix

$$A=\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix},$$

since a spacelike hyperbolic matrix has not an n-th root if n is even.

Theorem 5.5. *Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a lightlike hyperbolic real matrix whose polar representation is*

$$A=\mathrm{tr}A\begin{bmatrix} \frac{a}{\mathrm{tr}A} & \frac{b}{\mathrm{tr}A} \\ \frac{c}{\mathrm{tr}A} & \frac{d}{\mathrm{tr}A} \end{bmatrix}.$$

Then, the n-th root of the matrix A is

$$\sqrt[n]{A}=(\mathrm{tr}A)^{1/n}\begin{bmatrix} \frac{a}{\mathrm{tr}A} & \frac{b}{\mathrm{tr}A} \\ \frac{c}{\mathrm{tr}A} & \frac{d}{\mathrm{tr}A} \end{bmatrix}.$$

for $n\in\mathbb{Z}^+$.

Proof. It can be seen from Theorem 4.8 and Corollary 4.9. □

Remark 5.6. In the case n is even, n-th roots of A exist if and only if $\mathrm{tr}A>0$.

5.3. n-th Roots of a Parabolic Matrix

Theorem 5.7. *Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a\neq d$ be a timelike parabolic real matrix whose polar representation is*

$$A=\frac{\mathrm{tr}A}{2}\begin{bmatrix} \xi\theta+1 & \frac{2b\theta}{a-d}\xi \\ \frac{2c\theta}{a-d}\xi & 1-\xi\theta \end{bmatrix},$$

then the n-th roots of the matrix A are

$$\sqrt[n]{A} = \left(\tfrac{\mathrm{tr}A}{2}\right)^{1/n} \begin{bmatrix} \frac{\theta}{n}\xi + 1 & \frac{2b\theta}{n(a-d)}\xi \\ \frac{2c\theta}{n(a-d)}\xi & 1 - \frac{\theta}{n}\xi \end{bmatrix} \tag{5.6}$$

where $\theta = \frac{a-d}{|a+d|}$ and $\xi =$sign$(\mathrm{tr}A)$.

Proof. Let X be a matrix satisfying the equality $X^n = A$. Because of that A is a timelike parabolic matrix, X must be a parabolic matrix according to Corollary (4.13). Then, the matrix X can be in the form

$$X = \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \frac{\mathrm{tr}X}{2} \begin{bmatrix} \xi\beta + 1 & \frac{2y\xi\beta}{x-t} \\ \frac{2z\xi\beta}{x-t} & 1 - \xi\beta \end{bmatrix}.$$

where $\mathrm{tr}X = x + t$, $\beta = \frac{x-t}{|x+t|}$ and $(x-t)^2 = -4yz$. According to (4.7), we have

$$X^n = \left(\tfrac{\mathrm{tr}X}{2}\right)^n \begin{bmatrix} \xi n\beta + 1 & \frac{2y\xi n\beta}{x-t} \\ \frac{2z\xi n\beta}{x-t} & 1 - \xi n\beta \end{bmatrix}.$$

Thus, using the equations $(X^n)_{11} = A_{11}$ and $(X^n)_{22} = A_{22}$, we find

$$\begin{aligned} \left(\tfrac{\mathrm{tr}X}{2}\right)^n (\xi n\beta + 1) &= \tfrac{\mathrm{tr}A}{2}(\xi\theta + 1) \\ \left(\tfrac{\mathrm{tr}X}{2}\right)^n (1 - \xi n\beta) &= \tfrac{\mathrm{tr}A}{2}(1 - \xi\theta). \end{aligned}$$

Solving these two equations, we obtain $\frac{\mathrm{tr}X}{2} = \left(\frac{\mathrm{tr}A}{2}\right)^{1/n}$ and $\beta = \frac{\theta}{n}$. Therefore, we have

$$X^n = \left(\tfrac{\mathrm{tr}A}{2}\right) \begin{bmatrix} \xi\theta + 1 & \frac{2y\xi\theta}{x-t} \\ \frac{2z\xi\theta}{x-t} & 1 - \xi\theta \end{bmatrix}.$$

Also, we can see that the equality $X^n = A$ satisfies if and only if

$$\frac{y}{x-t} = \frac{b}{a-d} \text{ and } \frac{z}{x-t} = \frac{c}{a-d} \text{ and } \frac{x-t}{|x+t|} = \frac{a-d}{n|a+d|}.$$

According to this, we can obtain $y = bk$, $z = ck$, $x = \left(\frac{a+d}{2}\right)^{1/n} + \left(\frac{a-d}{2}\right)k$ and $t = \left(\frac{a+d}{2}\right)^{1/n} - \left(\frac{a-d}{2}\right)k$ where $k = \frac{1}{n}\left(\frac{a+d}{2}\right)^{\frac{1-n}{n}}$. As a result, we find

$$\sqrt[n]{A} = \left(\tfrac{\mathrm{tr}A}{2}\right)^{1/n} \begin{bmatrix} \frac{\xi\theta}{n} + 1 & \frac{2b\xi\theta}{n(a-d)} \\ \frac{2c\xi\theta}{n(a-d)} & 1 - \frac{\xi\theta}{n} \end{bmatrix}.$$

□

Remark 5.8. In the case n is even, n-th roots of A exist if and only if $\mathrm{tr}A > 0$.

Example 6. Let's find n-th roots of the parabolic matrix

$$A = \begin{bmatrix} 11 & -12 \\ 3 & -1 \end{bmatrix}.$$

The polar form of the matrix A is

$$A = 5 \begin{bmatrix} 1 + \theta & -2\theta \\ \theta/2 & 1 - \theta \end{bmatrix}$$

where $\theta = \dfrac{6}{5}$. Therefore, we get

$$\sqrt[n]{A} = 5^{1/n} \begin{bmatrix} 1+\theta/n & -2\theta/n \\ \theta/2n & 1-\theta/n \end{bmatrix}.$$

Theorem 5.9. *If* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be a timelike parabolic real matrix with* $a = d \neq 0$, *then* $\sqrt[n]{A}$ *is*

$$\begin{bmatrix} 1 & 0 \\ c/an & 1 \end{bmatrix}, \; a^{1/n} \begin{bmatrix} 1 & b/an \\ 0 & 1 \end{bmatrix} \text{ or } a^{1/n} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

according to $b = 0$, $c = 0$ *or* $b = c = 0$, *respectively. Moreover, if* n *is even and* $b = c = 0$, *then* $\sqrt[n]{A}$ *is*

$$\begin{bmatrix} t & s \\ \frac{a^{2/n}-t^2}{s} & -t \end{bmatrix}$$

for $t, s \in \mathbb{R}$, $s \neq 0$.

Proof. If A is a timelike parabolic matrix, then $bc = 0$. A can be one of the forms (3.5). So, according to Theorem 4.12, we find $\sqrt[n]{A}$ as

$$a^{1/n} \begin{bmatrix} 1 & 0 \\ c/an & 1 \end{bmatrix}, \; a^{1/n} \begin{bmatrix} 1 & b/an \\ 0 & 1 \end{bmatrix} \text{ or } a^{1/n} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

for $b = 0$, $c = 0$ or $b = c = 0$, respectively. Moreover, from the Theorem 4.5, we can find different roots for $A = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Assume that $X^n = A$ satisfies, where $X = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$. According to the equality

$$X^n = (-\det X)^{n/2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A,$$

we find $a = (-\det X)^{n/2}$ and $a^{2/n} = x^2 + yz$. Therefore, for $x = t$ and $y = s$, we obtain $z = \dfrac{a^{2/n} - t^2}{s}$. As a result,

$$\begin{bmatrix} t & s \\ \dfrac{a^{2/n} - t^2}{s} & -t \end{bmatrix}$$

is a n-th root of A for $t, s \in \mathbb{R}$, $s \neq 0$ and n is even. That is, in the case $b = c = 0$ and n is even, we have infinitely many n-th roots for A. □

Example 7. Let's find $\sqrt[6]{A}$ for the parabolic matrix

$$A = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}.$$

According to Theorem 5.9, we get

$$\sqrt[6]{A} = \begin{bmatrix} t & s \\ -\dfrac{1}{s}\left(t^2 - 2\right) & -t \end{bmatrix}$$

for $s, t \in \mathbb{R}$.

SOME CONCLUSIONS

1. Number of n-th roots of a 2×2 real matrix

As it can be seen from the above theorems, a 2×2 real matrix may have one, two, four, n or infinitely many n-th roots, or it may have not a root. We can summarize all results with the following corollary.

Corollary 5.10. *A matrix B is said to be an n-th root of a matrix A if $B^n = A$, where $n \geq 2$. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the number of n-th roots of a A is as follows :*

- *An elliptic matrix has exactly n n-th roots.*
- *A timelike hyperbolic matrix has exactly four n-th roots if n is even.*
- *A timelike hyperbolic matrix has only one n-th root if n is odd.*
- *A spacelike hyperbolic matrix has not an n-th root if n is even.*
- *A spacelike hyperbolic matrix has only one n-th root if n is odd.*
- *A lightlike hyperbolic matrix has only one n-th root if n is odd.*
- *A lightlike hyperbolic matrix has two n-th roots if n is even and $\mathrm{tr}A > 0$.*
- *A timelike parabolic matrix has only one n-th root if $a \neq d$ and $\mathrm{tr}A > 0$.*
- *A nonzero lightlike parabolic matrix has not an n-th root.*
- *A non-scalar timelike parabolic matrix has only one n-th root if $a = d$.*
- *A scalar matrix has infinitely many n-th roots.*
- *Each lightlike parabolic matrix is a root of zero matrix.*

2. De Moivre's Formula for 2D Rotation Matrices

The polar representations of the matrices

$$R_E = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad R_L = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}, \quad R_G = \begin{bmatrix} \theta+1 & -\theta \\ \theta & 1-\theta \end{bmatrix}$$

are themselves, since $\Delta_E = -\sin^2\theta$, $\Delta_L = \sinh^2\theta$, where R_E, R_L and, R_G are rotations matrices in the Euclidean, Lorentzian and Galilean plane, respectively. So, de Moivre's formula for these matrices as follows :

$$R_E^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}, \quad R_L^n = \begin{bmatrix} \cosh n\theta & \sinh n\theta \\ \sinh n\theta & \cosh n\theta \end{bmatrix}, \quad R_G^n = \begin{bmatrix} n\theta+1 & -n\theta \\ n\theta & 1-n\theta \end{bmatrix}.$$

for $n \in \mathbb{Z}$.

3. Comparing some methods for finding n-th degree roots of a 2×2 matrix.

Let's compare the practicability, the advantages and disadvantages of the methods used in the literature. And we explain why the method given in this study can be preferred to other methods.

- **Basic Algebraic Method :** Solving the system of higher degree equations can be difficult and messy.
- **Diagonalization :** It is a commonly used method. But, it works only for diagonalizable matrices.

- **Schur Decomposition Method :** It is works only for triangularizable matrices. A disadvantage of this method is that if A has nonreal eigenvalues, the method necessitates complex arithmetic if the root which is computed should be real.
- **Cayley Hamilton Method :** Even if it is the best method for finding square roots, it is difficult to apply for finding the roots of degree greater than 2. For higher degrees, the computations can become long and messy.
- **Newton Method :** It does not give results in rootless matrices. It is long and tedious for $n > 2$. If it is not known that matrix has not a root, it will be a complete waste of time.
- **Using Complex, dual and hyperbolic numbers :** It can only be used for some three specific matrix types.
- **Abel-Mobius Method :** It is the most difficult, tedious and complicated one in the methods.
- **De Moivre's Formula (Hybrid Numbers) :** This method can be used for all 2×2 matrices. It is suitable for the computer algorithm, and after type and character of the matrix is determined, the result can be directly calculated by substituting the n and θ values in the appropriate formula. No complicated function and process information is required. The basic matrix, trigonometry and hyperbolic function knowledge is sufficient to obtain the result. In addition to all of these, if the number n is greater than 2, operations are not confused and difficult. That is, in the case $n > 2$, it is an easy and fast alternative method that can be used to find n-th root of any 2×2 matrix. But, this method can be used only to find the real roots of a matrix.

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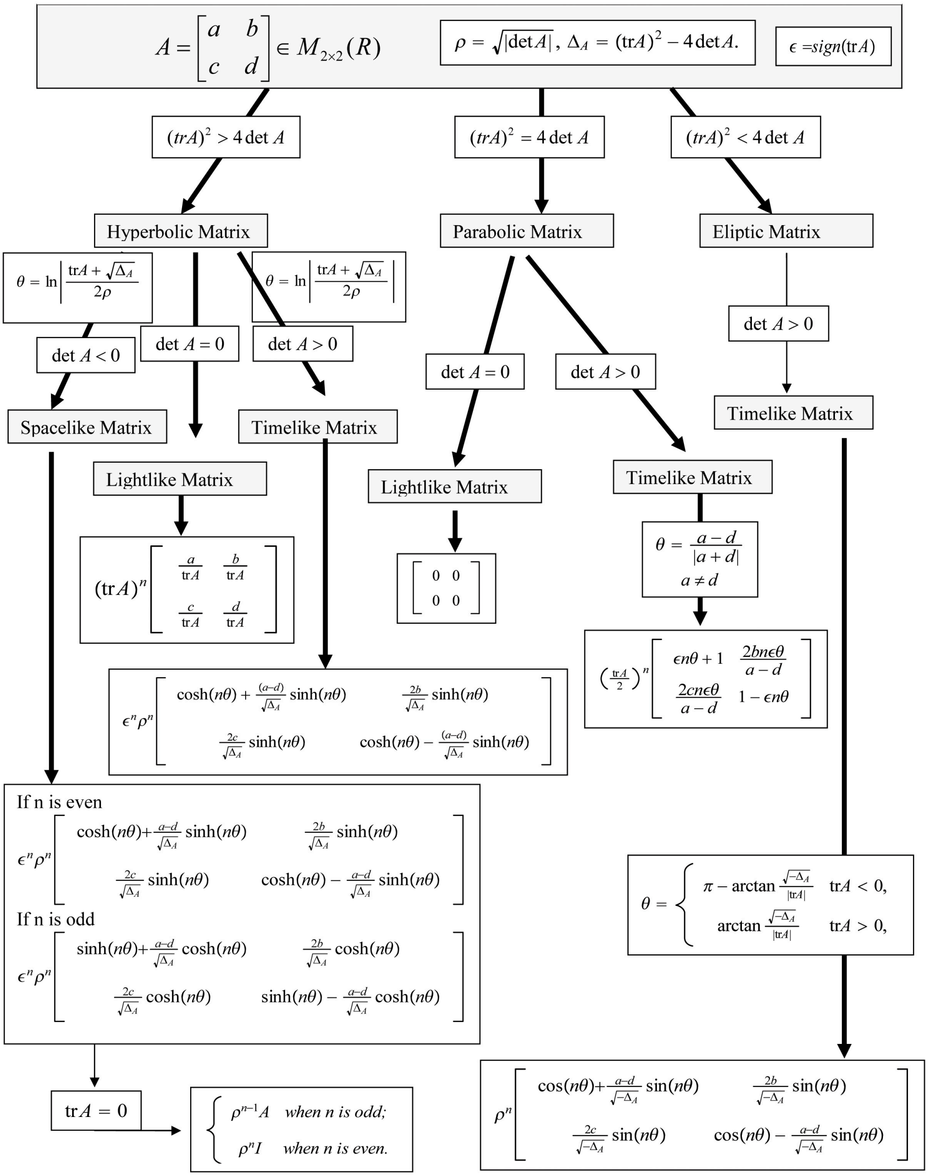
De MOIVRE'S FORMULAS FOR 2×2 REAL MATRİCES



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Example 8. Let's find n-th power of each of the following different types of matrices,

$$A=\begin{bmatrix} 3 & 4 \\ -2 & -1 \end{bmatrix}, \quad B=\begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix}, \quad C=\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix},$$
$$D=\begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix}, \quad E=\begin{bmatrix} 5 & 9 \\ -1 & -1 \end{bmatrix}, \quad F=\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}.$$

A is an elliptic matrix and polar form is

$$A=\sqrt{5}\begin{bmatrix} \cos(\theta)+\sin(\theta) & 2\sin(\theta) \\ \sin(\theta) & \cos(\theta)-\sin(\theta) \end{bmatrix}$$

where $\theta=\arctan 2$. Then, using the Theorem 4.1, we find that

$$A^n=\sqrt{5}^n\begin{bmatrix} \cos(n\theta)+\sin(n\theta) & 2\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta)-\sin(n\theta) \end{bmatrix}.$$

B is a timelike hyperbolic matrix, then using the Theorem 4.3, we get

$$B^n=(-1)^n 5^{n/2}\det\begin{bmatrix} \cosh n\theta+\frac{1}{2}\sinh n\theta & \frac{3}{2}\sinh n\theta \\ \frac{1}{2}\sinh n\theta & \cosh n\theta-\frac{1}{2}\sinh n\theta \end{bmatrix}$$

where $\theta=\ln\frac{\sqrt{5}}{5}$.

C is a spacelike hyperbolic matrix, then then using the Theorem 4.3, we find

$$C^n=\begin{bmatrix} \sinh n\theta+\frac{\sqrt{5}}{5}\cosh n\theta & \frac{2\sqrt{5}}{5}\cosh n\theta \\ \frac{2\sqrt{5}}{5}\cosh n\theta & \sinh n\theta-\frac{\sqrt{5}}{5}\cosh n\theta \end{bmatrix}$$

where $\theta=\ln(\sqrt{5}+2)$.

D is a lightlike hyperbolic matrix and the polar form is

$$D=8\begin{bmatrix} 3/4 & 3/8 \\ 1/2 & 1/4 \end{bmatrix}.$$

So, we get

$$D^n=8^n\begin{bmatrix} 3/4 & 3/8 \\ 1/2 & 1/4 \end{bmatrix}.$$

E is a parabolic matrix and polar form is

$$E=2\begin{bmatrix} 1+\theta & 3\theta \\ -\theta/3 & 1-\theta \end{bmatrix}$$

then, E^n is

$$E^n=2^n\begin{bmatrix} 1+n\theta & 3n\theta \\ -n\theta/3 & 1-n\theta \end{bmatrix} \text{ for } n\in\mathbb{Z}^+$$

where $\theta=3/2$. Finally, F is a lightlike parabolic matrix. Then,

$$F^n=0$$

for all $n\in\mathbb{Z}^+$.

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